Computer functions

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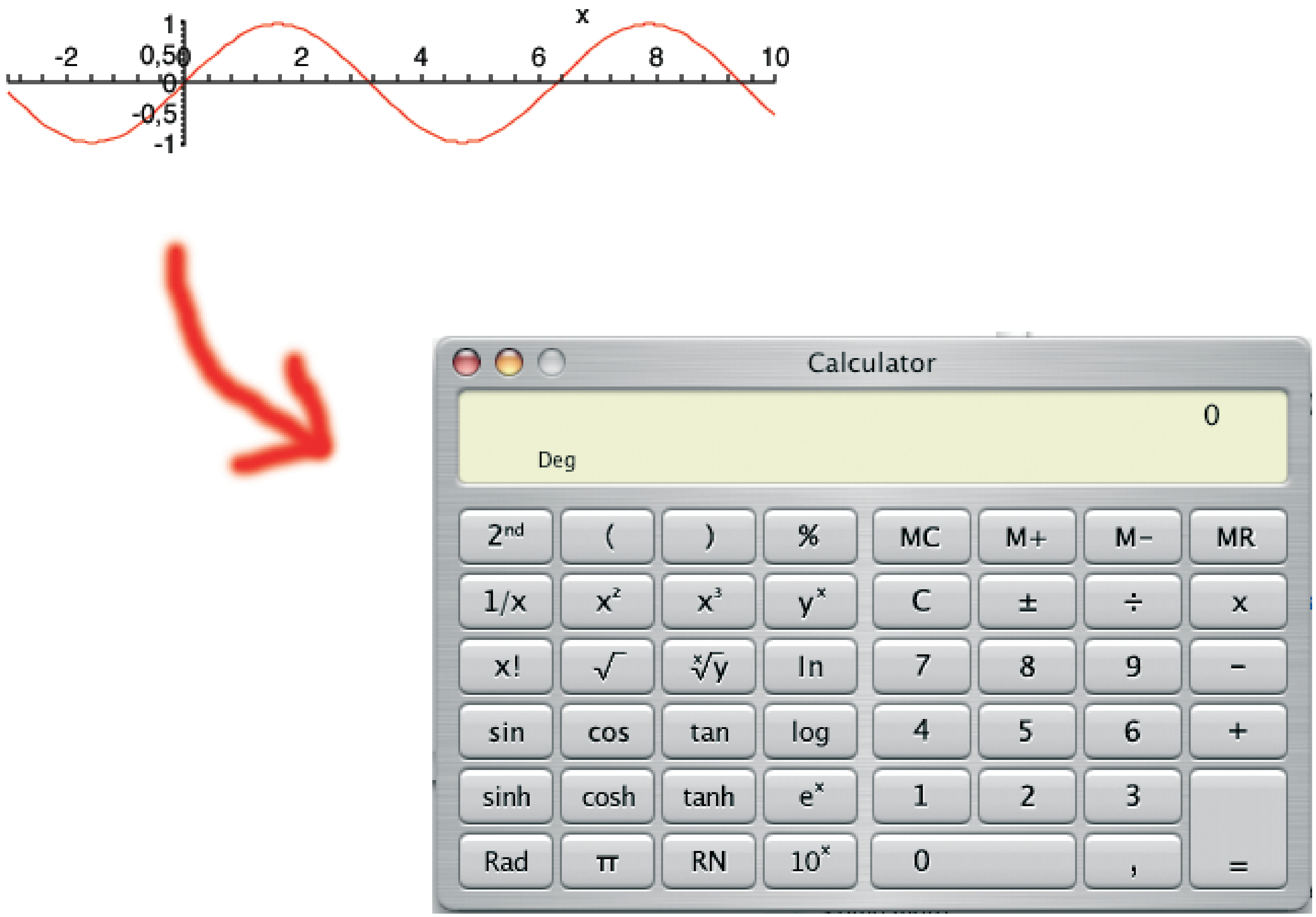
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We want to calculate the SINE:

1. Reducing the interval ($\mathbb{R}$, Float'Range,...) to $[-\frac{\pi}{2}, \frac{3\pi}{2}]$

2. Reducing the interval to $[-\frac{\pi}{2}, \frac{\pi}{2}]$

3. Reducing the interval to $[-\frac{\pi}{6}, \frac{\pi}{6}]$

4. Calculating the $\sin$e with ...

   (a) ... a Taylor Polynomial

   (b) ... a Chebyshev Polynomial

1. Reducing the interval to $[-\frac{\pi}{2}, \frac{3\pi}{2}]$

We need a function $y = f(x)$ with $0 \leq y < 2\pi$ (one period of sine) and

$$\sin x = \sin y.$$

So how can we do that? Why is this possible?

The function is called **entier**. It maps every $n \in \mathbb{R}$ to the biggest integer i with $i \leq n$. So our formula is

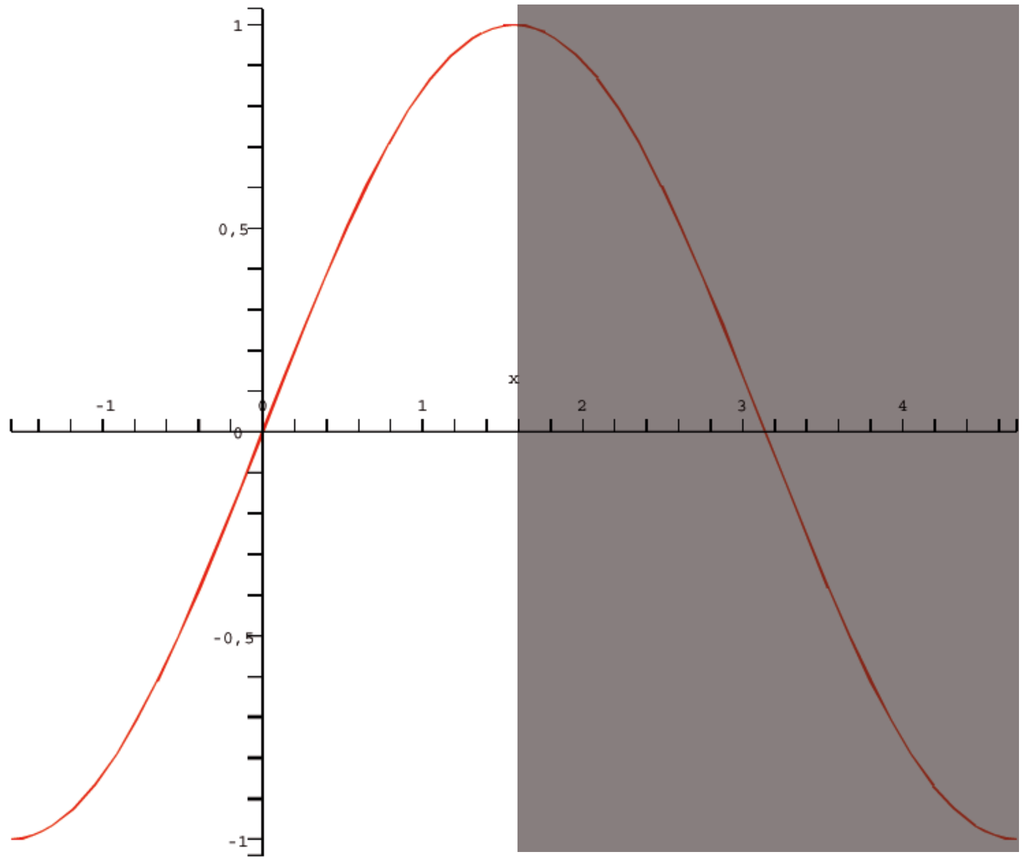
$$\sin y = \sin(x - 2\pi * \text{entier}\frac{x}{2\pi}) = \sin x.$$

As it is much better to have the interval $[-\frac{\pi}{2}, \frac{3\pi}{2}]$ instead of $[0, 2\pi]$, we change the above line to

$$\begin{aligned} \sin y &= \sin(x - 2\pi * \text{entier}\frac{x + \frac{\pi}{2}}{2\pi}) \\ y &= (x - 2\pi * \text{entier}\frac{x + \frac{\pi}{2}}{2\pi}). \end{aligned}$$

2. Reducing the interval to $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Now we have the sine on the interval $[-\frac{\pi}{2}, \frac{3\pi}{2}]$. The sine is symmetric to $y = \frac{\pi}{2}$ so we can reduce the interval to $[-\frac{\pi}{2}, \frac{\pi}{2}]$ (next slide).



1
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-0,5
-1
-1
0
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2
3
4
x

The formula for that is

$$z := \begin{cases} y & \text{if } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \\ \pi - y & \text{if } \frac{\pi}{2} < y \end{cases}$$

To calculate the $\sin$e in in the interval we will use a **Taylor Polynominal**. But before, we will reduce the interval some more.

3. Reducing the interval to $[-\frac{\pi}{6}, \frac{\pi}{6}]$

From the addition theorems we get the following term:

$$\sin(3u) = 3\sin u - 4\sin^3 u.$$

With $z = 3u$ and some small changes we get:

$$\sin z = 3(1 - \frac{4}{3}\sin^2\frac{z}{3})\sin\frac{z}{3}.$$

This can easily be done with the two addition theorems and the Pythagoras

$$\begin{aligned} \sin(a+b) &= \sin a \cos b + \sin b \cos a \\ \cos(a+b) &= \cos a \cos b - \sin a \sin b \\ 1 &= \sin^2(x) + \cos^2(x) \end{aligned}$$

$$
\begin{aligned}
\sin(3u) &= \sin(u+2u) \\
&= \sin u \cos 2u + \sin 2u \cos u \\
&= \sin u(\cos(u+u)) + \sin(u+u)\cos u \\
&= \sin u(\cos^2 u - \sin^2 u) + (2\sin u \cos u)\cos u \\
&= \sin u \cos^2 u - \sin^3 u + 2\sin u \cos^2 u \\
&= \sin u(3\cos^2 u - \sin^2 u) \\
&= \sin u(3(1-\sin^2 u) - \sin^2 u) \\
&= \sin u(3 - 3\sin^2 u - \sin 2u) \\
&= 3\sin u - 4\sin 3u
\end{aligned}
$$

4. (a) Calculating the $\sin$ with a Taylor Polynomial

Now we can calculate the $\sin$ with a Taylor Polynomial (for example). This can now be done with very little effort since we are close to 0 (we take $x_0 = 0$, then we get an odd function).

A Taylor Polynomial of degree n is defined

$$\begin{aligned} T_n(x) &= \sum_{k=0}^{n} \frac{1}{k!} * f^{(k)}(x_0) * (x - x_0)^k \\ f(x) &= T_n(x) + R_n(x, \xi) \\ R_n(x, \xi) &= \frac{1}{(n+1)!} * f^{(n+1)}(\xi) * (x - x_0)^{n+1}. \end{aligned}$$

Since we do this for a calculator we want to have 7 digits accuracy. So $R_n(x, \xi)$ has to be $< 0.5 * 10^{-7}$.
Our $x_0 = 0$, since it is the centre of our interval, $f(x) = \sin x$, and we try it for $n = 8$.

$$
\begin{aligned}
T_n(x) &= \sum_{k=0}^{n} \frac{1}{k!} * f^{(k)}(x_0) * (x - x_0)^k \\
T_8(u) &= \sum_{k=0}^{8} \frac{1}{k!} * f^{(k)}(0) * u^k \\
&= \frac{1}{1!} * f^{(1)}(0) * u + \frac{1}{2!} * f^{(2)}(0) * u^2 \\
&\quad + \frac{1}{3!} * f^{(3)}(0) * u^3 + \frac{1}{4!} * f^{(4)}(0) * u^4 + \ldots \\
&= 1 * 1 * u + \frac{1}{2!} * 0 * u^2 + \frac{1}{3!} * (-1) * u^3 + \frac{1}{4!} * 0 * u^4 + \ldots \\
&= u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!}
\end{aligned}
$$

$$
= u - 0.1666666667u^3 + 0.008333333333u^5 - 0.0001984126984u^7
$$

$$
\begin{aligned}
R_n(x,\xi) &= \frac{1}{(n+1)!} * f^{(n+1)}(\xi) * (x - x_0)^{n+1}. \\
|R_8(u,\xi)| &= |\frac{1}{9!} * \sin^9(\xi) * (u)^9| \\
&= \frac{|\cos(\zeta)| * |(u)^9|}{9!} \leq \frac{|u|^9}{9!} \\
\text{relative error} &= \frac{|p(u) - \sin(u)|}{|sin(u)|} \\
&= \frac{|R_8|}{|\sin(u)|} \leq \frac{|R_8|}{0.95|u|} \leq \frac{1.1|u|^8}{9!} \\
&\leq \frac{1.1}{9!}(\frac{\pi}{6})^8 \approx 0.18 * 10^{-7} < 0.5 * 10^{-7}
\end{aligned}
$$

Now we have to write that algorithm optimized for the computer:

$$
\begin{aligned}
\tilde{x} &= x * 0.15915494 \\
\tilde{y} &= \tilde{x} - \text{entier}(\tilde{x} + 0.25) \\
\tilde{z} &= \begin{cases} \tilde{y} & \text{, if } -0.25 \leq \tilde{y} \leq 0.25 \\ 0.5 - \tilde{y} & \text{, if } 0.25 < \tilde{y} \end{cases} \\
v &= \tilde{z}\tilde{z} \\
w &= \tilde{z}(3.32464499 + v(-2.43058747 \\
&\quad + v(0.53308748 - v * 0.0556757))) \\
\sin x &= w(1.88988158 - ww)
\end{aligned}
$$

We have reduced the sine to 8 multiplications and 7 additions.

4. (b) Calculating the sine with a Chebyshev Polynomial

There are other ways then using a Taylor Polynomial. For example can a Chebyshev Polynomial be used, this is better since with the same number of terms it is more accurate. Since this is a much more complicated thing, we have to learn some more basics.

Optimal polynomial approximations

We want to approximate a function $F(x)$ in the closed interval $[a,b]$ by means of a polynomial of degree $\leq n$.

1. the polynomial $P_n(x)$ of degree $\leq n$ for which $\max_{[a,b]} |P_n(x) - F(x)|$ is as small as possible, if it is absolute error that we are interested in (*minimax-absolute-error*), or

2. the polynomial $P_n(x)$ of degree $\leq n$ for which $\max_{[a,b]} |\frac{P_n(x)-F(x)}{F(x)}|$ is as small as possible, if it is relative error that we are interested in (*minimax-relative-error*).

Chebyshev's theorem on polynomial approximations

Let $u(x)$ denote a function continuous in a closed, finite interval $[a, b]$, and let $v(x)$ denote a function continuous and nonzero in $[a, b]$. Let V_n denote the set of polynomials of degree $\leq n$. There exists a unique polynomial $P_n^*(x)$ in V_n such that

$$\max_{[a,b]} \left| \frac{P_n^*(x)}{v(x)} - u(x) \right| = \min_{P_n(x) \text{in} V_n} \max_{[a,b]} \left| \frac{P_n(x)}{v(x)} - u(x) \right|.$$

. . .

. . .

Let $P_n(x)$ denote a polynominal in V_n. Then $P_n(x)$ is $P_n^*(x)$ if and only if there exist $N \geq n+2$ points in $[a,b]$,

$$x_1^* < x_2^* < x_3^* < \cdots < x_N^*$$

such that

$$\frac{P_n(x_k^*)}{v(x_k^*)} - u(x_k^*) = (-1)^k \mu^*$$

$$k = 1, 2, 3, \ldots, N,$$

where

$$|\mu^*| = \max_{[a,b]} \left|\frac{P_n(x)}{v(x)} - u(x)\right|.$$

A proof of this theorem is beyond the scope of this lecture. Listeners wishing to study the proof can find one in Achieser (1956)*.

Two ways of construing the foregoing theorem are of interest to us:

*Achieser, N. I. (1956): *Theory of Approximation*. Ungar, New York. english translation by C. J. Hyman.

1. With $v(x) = 1$ and $u(x) = F(x)$, the function $\frac{P_n(x)}{v(x)} - u(x)$ becomes the **absolute-error function** $P_n(x) - F(x)$. In this case, the theorem asserts that there exists a unique polynimial $P_n^*(x)$ of degree $\leq n$ that approximates $F(x)$ with **minimal absolute error** in $[a, b]$. The theorem further asserts that $P_n^*(x)$ is uniquely characterized by the fact that the absolute-error finction $P_n*(x) - F(x)$ possesses at least $n+2$ extreme points in $[a, b]$ at which it is alternately positive and negative and at which the magnitudes of $P_n^*(x) - F(x)$ are equal.

2. With $v(x) = F(x)$ and $u(x) = 1$, where now it is assumed that $F(x) \neq 0$ in $[a, b]$, the function $\frac{P_n(x)}{v(x)} - u(x)$ becomes the **relative-error function** $\frac{P_n(x) - F(x)}{F(x)}$. In this case the theorem asserts that there exists a unique polynomial $P_n^*(x)$ of degree $\leq n$ that approximates $F(x)$ with **minimax relative error** in *[a, b]*. This $P_n^*(x)$ is uniquely characterized by the fact that the relative-error function $\frac{P_n^*(x) - F(x)}{F(x)}$ posesses at least $n + 2$ extreme points in $[a, b]$ at which it is alternatively positive and negative and at which the magnitude of $\frac{P_n^*(x) - F(x)}{F(x)}$ are equal.

An argument for which the maximum magnitude of the error function is attained is called *critical point*, of the approximation. A minimax polynomial approximation to a function is specifically associated with an integer n and an approximation interval $[a, b]$. Generally, there is also a difference between the the function with *minimax-absolute-error* and the one with *minimax-relative-error*.

We want a a function of the degree 8 in the interval $[-\frac{\pi}{6}, \frac{\pi}{6}]$ and we want minimax-absolute-error.

Remez' method for polynomial approximations

This is one of two methods by E. Ya. Remez and is called *Remez' second method*.
We want to approximate $F(x)$ in $[a,b]$ and want to determine the polynomial $P_n^*(x)$ of the degree $\leq n$ that approximates $F(x)$ with minimax absolute error in $[a,b]$.
Let $P_n^*(x) = a_0^* + a_1^* x + \cdots + a_n^* x^n$. For the sake of simplicity, we assume that $P_n^*(x) - F(x)$ is a standard error function. Then $P_n^*(x) - F(x)$ possesses exactly $n+2$ critical points in $[a,b]$, including a and b. Let these be denoted by $x_k^*, k = 1, 2, \ldots, n$ and labelled so that

$$a = x_1^* < x_2^* < x_3^* < \cdots < x_{n+2}^* = b.$$

Then we know by chebyshev's theorem that

$$a_0^* + a_1^* x_k^* + \cdots + a_n^* (x_k^*)^n - F(x)_k^* = (-1)^k \mu^*,$$
$$k = 1, 2, \ldots, n+2,$$

where

$$|\mu^*| = \max_{[a,b]} |P_n^*(x) - F(x)|.$$

The objective in this method is to compute iteratively the x_s^*'s, μ's, andthe coefficients of $P_n^*(x)$.

1. Initially select $n+2$ numbers $x_k, k = 1, 2, \ldots, n+2$, such that

$$a = x_1^* < x_2^* < x_3^* < \cdots < x_{n+2}^* = b.$$

2. Compute the coefficients of a polynomial $P_n(x) = a_0 + a_1 x + \cdots + a_n x^n$ and the number μ by solving the system of $n+2$ linear equations

$$a_0 + a_1 x_k + \cdots + a_n (x_k)^n - (-1)^k \mu = F(x_k), \qquad k = 1, 2, \ldots, n+2,$$

for $n+2$ unknowns $a_0, a_1, \ldots, a_n$, and μ.

3. Locate the extreme points in $[a, b]$ of the absolute-error function $P_n(x) - F(x)$. For the sake of simplicity, we assume that there are exactly $n+2$ extreme points, including a and b. Let these be labelled $y_k, k = 1, 2, \ldots, n+2$, where

$$a = y_1 < y_2 < y_3 < \cdots < y_{n+2} = b.$$

4. Replace x_k with y_k for $k = 1, 2, \ldots, n+2$, and repeat the sequence of steps given above beginning with step (2).

x_k converges to x_k^*, a_k converges to a_k^*, and μ converges to μ^*. The convergence is quadric. A good algorithm to compute the starting values in step 1 is

$$x_k = \frac{1}{2}\cos\frac{(n-k+2)\pi}{n+1} + \frac{1}{2}(b+a),$$
$$k = 1, 2, \ldots, n+2.$$

If F(x) is an even or an odd function and the interval is of the form $[-a, a]$, you can use $[0, a]$ instead.

And now we will have fun using Maple.

Enjoy!