Seminar: Interplay to numerical simulation and modeling

Symbolic Analisys of Dynamical Systems

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Abstract

In this talk we present the method of numerical investigating of discret dynamical systems called symbolic analisys. Consider a finite covering of the space and identify points lying in same cells of this covering. Such point of view fully corresponds with a situation in real measurings — devices show us only finite number of digits. Under this assumption we consider a graph, which vertecies corresponds cells of covering and edges shows that point from first cell can go to second cell. Paths in this graph correspond to trajectories of initial dynamical system. In this talk presented some theorems and algorithms based on this conception. An example of calculation of topological entropy is considered .

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1 Introduction

I'd like to tell you about Dynamical Systems and one of the methods of their investigation. Why are these systems so important for us? Why is it worth knowing about such systems? Such systems can be found everywhere around us. Any object, for which the notion of state and the law of evolution are determined, can constitute an example of a dynamical system. Pendulum, "prey-predator" system, three bodies' problem and many others are all examples of such systems.

Let us introduce the notion of a dynamical system. Consider a homeomorphism f of some space M. It defines a dynamical system on M.

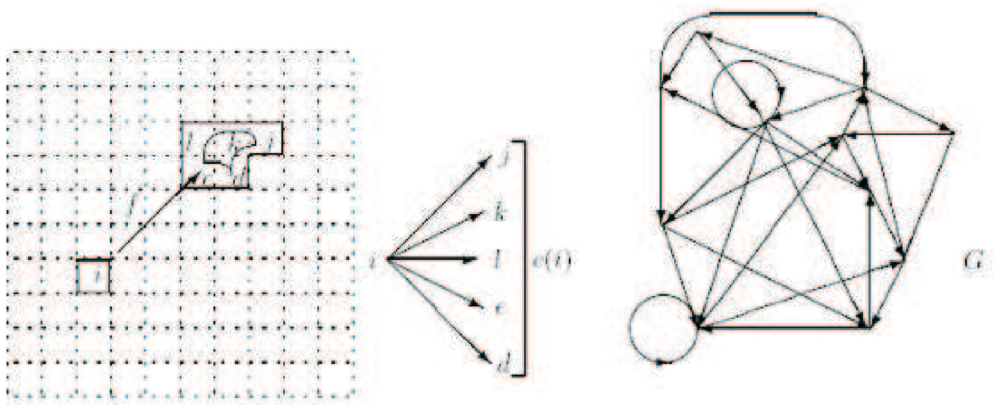
Definition 1. For a point $x \in M$ its trajectory is the set $\dots x_{-1} = f^{-1}(x), x_0 = x, x_1 = f(x), x_2 = f^2(x), \dots$.

Trajectories may be of one of three types.

1. Fixed points. When all points in the trajectory are the same. They are corresponding to the constant processes.
2. Periodical points. If for some n holds $f^0(x) = f^n(x)$. Then $f^k(x) = f^{n+k}(x)$. These points are corresponding to the periodical nature processes
3. All the other points.

2 Symbolic Image

We'd consider a dynamical system on a compact manifold M, generated by a homeomorphism f and a finite covering of space M by closed sets $C = \{M(1), \dots, M(n)\}$. Let us determine a graph G which vetecies "i" corresponds to an element of the covering $M(i)$. Let's then connect vertices "i" and "j" by a directed edge if there exists a point in $M(i)$, the image of which lies in $M(j)$ or, in a more formal language, $f(M(i)) \bigcap M(j) \neq 0$. This graph is called the symbolic image of the dynamical system.



Sequence of vertices (finite or infinite) $\ldots, i_0, \ldots, i_n \ldots$ such that for any k vertices i_k and i_{k+1} are connected by an edge is called a path in the graph. Note that to each trajectory of dynamical system corresponds a path in a symbolic image (but not conversely).

3 Periodic points

In the dynamical systems theory an important role is played by periodic points. They correspond to periodic nature processes. We'll describe the method of searching the periodical points by the methods of symbolic analisys.

A path $i_1, \ldots, i_k$ in graph G is called periodic if $i_1 = i_k$. Each periodic point of the dynamical system corresponds to a periodic path in the symbolic image. So the periodic points may lie only in such cells $M(i)$ that the vertex "i" in the symbolic image lies in some periodic path.

Below we present an algorithm based on this fact. It allows finding of all p-periodic points.

1. Consider any starting covering C. Construct its symbolic image G. Let the maximum diameter of cell be d_0.
2. Find all periodical points of the graph G with period p. Consider the union of elements of the fragmentation $P = \bigcup M(i_k)$, such that i_k is a p-periodic point of the graph G (i.e. point that lies in some periodic path, length p). This union is a close neighborhood of the set of p-periodic points.
3. Consider a subdivision of cells from P. Make their diameters smaller at least in 2 times. By this way we create a new covering.
4. Construct the symbolic image f of the new covering. It is easy to see that we need the symbolic image only for cells from P, because other cells don't contain periodic points.

5. Go to step 2.

It is simple to show that following statement holds [1]

Theorem 1. *In terms of above algorithm* $\bigcap_{k=1}^{\infty} P_k = Per(p)$, *where* $Per(p)$ *is the set of* p*-periodic points of the dynamical system.*

So, by increasing the number of iterations we may get an approximation of the set of the p-periodic points with any given precision. But at each step we can't guarantee the existence of p-periodic points in cells from P_k. In real problems the symbolic dynamics is usually applied to get starting approximations for more precise algorithms. For example, the Newton Method is applied [1], [2]

4 Entropy

Now, when we presented a basic example of symbolic dynamics application, let's consider a more difficult case.

Why are dynamical systems, so simply defined, being studied so intensively and still have open problems? The reason is that even very simple dynamical systems may have very complex structure of trajectories.

Let's look at one of the notions employed to estimate the complexity of this structure numerically.

4.1 Definition of topological entropy

Let M be a compact metric space. Consider a discrete dynamical system generated by homeomorphism $f : M \to M$. And let $C = \{M(1), \ldots, M(n)\}$ – be a finite open covering of M. Consider a trajectory of point x of length $N : x_k = f^k(x), \quad k = 0, \ldots, N-1$ and its code, i.e. the sequence $\xi(x) = \{i_k, k = 0, \ldots, N-1\}$, where $x_k \in M(i_k)$. Terms of this sequence are corresponding to the cells in which the trajectory of x lies. In this case we will say that the sequence $\xi = \{i_k\}$ is an admissible code. It is easy to see that not all of sequences with $\{i_k\}$ indices are admissible codes. For measuring the growth number of admissible paths, depending from the length of path N one usually uses the value $h = \lim_{N\to\infty} \frac{\log_a K(N)}{N}$, where $K(N)$ is the number of different admissible codes of length N. Base of logarithm may be any digit greater than 1, usually $a = 2$ or $a = e$. It was found that $h = 0$ for simple systems and $h > 0$ for systems with chaotic behavior. In the second case we may approximate $K(N) = Ba^{hN}$, where B is a constant, i.e. the number of admissible paths, growing like the exponent with index h. The value h is considered as a measure of a chaos degree within the system.

Why exactly is this character accepted as a measure of chaos? Assume that we know the trajectory's code of length N and we want to know, which code will have the next p trajectory's elements. How many different possibilities of such a prolongation are there? In average, this number will equal to $K(N+p)/K(N)$. Assume $h > 0$. Then $K(N)$ grows exponentially, i.e. $K(N) = Ba^{hN}$, so $K(N+p)/K(N) \approx a^{hp}$ i.e. for any large N it will be separated from 1. It means that we can't find an approximated prolongation for any longer part of trajectory's code. We would not have such an effect in case of polynomial dependency, where $K(N) = AN^{\alpha}$, then $K(N+p)/K(N) \approx (1+p/N)^{\alpha}$, that tends to 1 as $N \to \infty$.

Let's introduce the exact mathematical notion of topological entropy.

Let a homeomorphism $f : M \to M$ be given and let $C = \{M(1), \dots, M(n)\}$ be a finite open covering of M. For any $M(i_0)$, let's find the cell $M(i_1)$ of covering C, for which its intersection with preimage of $M(i_0)$ isn't empty: $M(i_0) \bigcap f^{-1}(M(i_1)) \neq 0$. Further, find $M(i_2)$, such that $M(i_0) \bigcap f^{-1}(M(i_1)) \bigcap f^{-2}(M(i_2)) \neq 0$ and so on. In other words, consider such cell of covering $M(i_k)$, where $0 < i_k < n$, for which $\bigcap_{k=0}^{N-1} f^{-k}(M(i_k)) \neq 0$.

Denote this set by $M(i_0 i_1 \dots i_{N-1})$, and their aggregate by C^N . It is easy to see that for $x \in \bigcap_{k=0}^{N-1} f^{-k}(M(i_k))$, holds $f^k \in M(i_k)$ for $k = 0, \dots, N-1$. So, for each x from $M(i_0 i_1 \dots i_{N-1})$ its $N-1$ iteration passes the set $\bigcup M(i_k), \quad k = 0, 1, 2, \dots, N-1$. The sets C^N form a finite covering of M too. Since the sets $M(i)$ may intersect each other, the elements of C^N may intersect each other too. Denote by $\rho(C^N)$ the number of sets of minimal subcovering that may be chosen from C^N. We may say that $\rho(C^N)$ is number of different codes of trajectories of length N. Suppose

$$h(C) = \lim_{N \to +\infty} \frac{\log \rho(C^N)}{N}.$$

Definition 2. The value $h(f) = \sup_C h(C)$ where supremum is taken over all open coverings, is called the *topological entropy* of the mapping f.

Now let's show difference between these two definitions. Consider the real line and the dynamical system, generated by the identical mapping. Cover real line with intervals. All trajectories will be fixed points. Look at points lying in the intersection of neighbour intervals. By the first definition any sequence of these two intervals will be an admissible code. So there will be at least $n * 2^N$ admissible codes of length N where n is a number of intervals. This means that entropy will be 1 or higher. By the second definition $M(i_0 i_1 \dots i_{N-1})$ may be only an interval or intersection of 2 intervals. So the initial covering will be a subcovering of C^N it means that $\rho(C^N) \leq n$ and the entropy will be 0.

Unfortunately, this exact mathematical definition isn't constructive. Let's introduce two new notions.

Definition 3. We will say that covering D is inscribed into covering C and write $D \succ C$ if each element of covering D is contained in some element of covering C

Definition 4. Call a sequence of coverings C_n exhaustive if for each covering B there exists n^* such that $C_n \succ B$ for any $n > n^*$

Theorem 2. *If C_n is exhaustive then $h(f) = \lim_{n\to\infty} h(C_n)$.*

This theorem opens a constructive way for calculation of entropy.

4.2 Calculation of topological entropy

Definitions

Let's start this section with some definitions and theorems.

Definition 5. Let M be a compact space. Let's call $C = \{M(1), \dots, M(n)\}$ a closed finite covering for M if the following requirements hold:

1. $\bigcup M(i) = M$;
2. each cell is the closure of its interior $M(i) = Cl(Int(M(i)))$;
3. cells intersect by their boundaries only: $M(i) \bigcap M(j) = \partial M(i) \bigcap \partial M(j)$ for $i \neq j$.

The following theorem shows that such coverings may be useful in estimation of entropy.

Theorem 3. *If C_k is a sequence of closed finite coverings of M such that $C_{k+1} \succ C_k$ and $diam(C_k) \to 0$ than*

1. $h(C_k) \leq h(C_{k+1})$,
2. $h(f) \leq \lim h(C_k)$.

Now let's introduce the defininition that will be useful in future.

Definition 6. Consider a finite number of symbols $a_1, \dots, a_n$ and some set P of sequences of these symbols. Define the entropy of this set $h(P)$ by $h(P) = \lim \log K(N)/N$, where $K(N)$ is the number of sequences with length N into P.

Subdivision process

Describe subdivision process. Let $C = \{M(i)\}$ be a closed finite covering of M and let G_1 be its symbolic image. Let D be a subdivision of C and G_2 its symbolic image. Denote by $m(i,k)$ cells of D, corresponding to the cell $M(i)$ of C, and (i,k) vertices of G_2. Determine mapping $s : G_2 \to G_1$

that will map vertices (i,k) into vertex i. It is easy to see that this mapping maps edges into edges. Thus the graph G_2 is mapped into subgraph of G_1.

Let's define the space of vertices of graph G as a set of sequences of vertices, such that every two neighbouring vertices are connected by the edge.

$$P_G = \{\xi = \{v_i\} : v_i \text{ connected to } v_{i+1}\}$$

For the sake of simplicity such sequences will be further referred to as paths.

Given such definition, we may extend a map s to P_2 and P_1 spaces. Denote $s(P_2) = P_1^2$. It is easy to see that $P_1^2 \subset P_1$ and that no real trajectory corresponds to any of sequences from $P_1 \setminus P_1^2$. Indeed, if we had a real trajectory x_n it would have a representation in P_2, such that the initial path is an image of this representation. Contradiction.

Theorem 4. *Following inequations holds*

1. $h(P_1^2) \leq h(P_1)$,
2. $h(P_1^2) \leq h(P_2)$.

Consider sequence of inscribed closed coverings $C_0, C_1, \ldots, C_k, \ldots$ and their corresponding graphs $G_0, G_1, \ldots, G_k, \ldots$. Than it is easy to see from previous considerations that the diagram is commutative

$$\begin{tikzcd}
V_0 & V_1 \arrow[l, "s_0"'] \arrow[d, "G_1"'] & V_2 \arrow[l, "s_1"'] \arrow[d, "G_2"'] & \cdots \arrow[l, "s_2"'] & M \arrow[l, "s"'] \arrow[d, "f"] \\
V_0 \arrow[u, leftarrow, "G_0"] & V_1 \arrow[l, "s_0"'] & V_2 \arrow[l, "s_1"'] & \cdots \arrow[l, "s_2"'] & M \arrow[l, "s"']
\end{tikzcd}.$$

Here under the term "commutative" we mean that $s_i G_{i+1}(z) \subset G_i s_i(z)$. In this case we have sequence of admissible paths and mappings between them

$$P_0 \overset{s_0}{\leftarrow} P_1 \overset{s_1}{\leftarrow} P_2 \overset{s_2}{\leftarrow} \ldots \overset{s}{\leftarrow} \{T_f\},$$

Where $\{T_f\}$ is space of trajectories of f.

Now we can construct the following sets of paths:

$$
\begin{aligned}
P_l^{l+1} &= s_l(P_{l+1}) \\
P_l^{l+2} &= s_l s_{l+1}(P_{l+2}) \\
\dots & \\
P_l^{l+r} &= s_l s_{l+1} \dots s_{l+r-1}(P_{l+r}), \\
\dots & \quad .
\end{aligned}
$$

So we have double sequence of paths P_l^k where $k > l$ and corresponding sequence of entropies $h_l^k = h(P_l^k)$.

The next theorem shows us the constructive way of calculating the topological entropy.

Theorem 5. *Let $C_1, C_2, \dots, C_k, \dots$ will be as above, a sequence of embedded closed coverings of M, such that $diam(C_k) \to 0$, then for each l holds*

1. *$P_l^k \supset P_l^{k+1}$ for each $k > l$ and entropy h_l^k decreases by k : $h(P_l^k) \geq h(P_l^{k+1})$.*
2. *Set of coded trajectories $Cod_l = \bigcap_{k>l} P_l^k$.*
3. *$h_l = h(Cod_l) = \lim_{k\to\infty} h_l^k$ and h_l grows by l.*
4. *If f is a Lipshitch's mapping then sequence h_l has a finite limit h^* and $h(f) \leq h^*$.*

Since to estimate the topological entropy we should calculate values of h_k^l. We can construct sets P_l^k thus we may calculate these values. For detailed algorithm we recommend [3].

4.3 Example

Now let's look at the results of application of this algorithm. Consider Henon's mapping

$$f(x, y) = (1 - 1.4x^2 + 0.3y, x)$$

into part of the plane [-1.5, 1.5]x[-1.5, 1.5].

Let's take a sequence of embedded coverings

$$C_k = \left\{ \left[-1.5 + \frac{3l}{2^k}, -1.5 + \frac{3(l+1)}{2^k} \right] \times \left[-1.5 + \frac{3p}{2^k}, -1.5 + \frac{3(p+1)}{2^k} \right] \right\},$$

where $k \in \mathbf{N}, \quad l, p = 0, \dots, 2^k - 1.$

We may apply the described algorithm for this mapping. The following table shows the result:

k	$\lvert \mathrm{Ver}_k \rvert$	$\lvert E_k \rvert$	h_{1k}	h_{2k}	h_{3k}	h_{4k}	h_{5k}
1	4	10	-	-	-	-	-
2	12	54	0.85054	-	-	-	-
3	34	174	0.81455	1.06881	-	-	-
4	78	505	0.69273	0.89793	1.13639	-	-
5	181	1192	0.60009	0.76660	0.96874	1.15955	-
6	453	2885	0.57082	0.65094	0.81607	0.96145	1.15546
7	1108	7216	0.53494	0.57784	0.69467	0.80663	0.97145
8	2588	17836	0.50242	0.52710	0.61530	0.69029	0.81573
9	5915	42069	0.48540	0.49440	0.54327	0.59310	0.69123
10	13338	96921	0.47584	0.47751	0.50856	-	-
11	31534	218644	0.46761	-	-	-	-

We may see the convergence to $h^* = 0.46 + \varepsilon$. It is known from employing other techniques that $h(f) = 0.4651$ [4]. So our algorithm allow us to get good results.

5 Conclusion

We have described a method of symbolic analysis of dynamical systems. The main idea of this method lies in the substitution of a continuous object "dynamical system" by a discrete object — a graph. The main problem of this method is that a considerable amount of information about the non-local behavior is lost. But we have a methodology of solving this problem that was described in the second part.

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