Cryptography and Elliptic curves

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Outline

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A **key** is a secret parameter for the cipher algorithm

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- private key cannot be practically derived from public key

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The main idea: security is based on the computational complexity of "hard" problems:

- integer factorization problem
- discrete logarithm problem

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- **DSA (Digital Signature Algorithm)** was developed in 1991 and is related to the discrete logarithm problem
- **ECDSA (Elliptic Curve Digital Signature Algorithm)** is a modification of DSA involving elliptic curve groups, which was proposed in 1992 by Scott Vanstone. It provides smaller key sizes for the same security level and that's why it has become the most popular digital signature.

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Remark: Public-key systems are computationally expensive $\Rightarrow$ in practice, a message is hashed (using a **cryptographic hash function**) and the smaller "hash value" is signed $\Rightarrow$ a receiver computes the hash of the message himself and verifies it.

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Candidates:

- a product of two large primes (RSA)
- an exponentiation in the finite field (DSA, ECDSA)

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Remark: No efficient algorithm for computing discrete logarithms is known $\Rightarrow$ discrete exponentiation is a candidate for **one-way function**

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- (r, s) is a **signature** for the message m

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- Accept the signature $\Leftrightarrow v = r$

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$$\boxed{r = (g^k \ mod \ p) \ mod \ q = (g^{u_1} y^{u_2} \ mod \ p) \ mod \ q = v}$$

The algorithm always accepts the **true** signatures

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For **elliptic curve cryptography** we need one of two cases:
$q = p$, where p is an odd prime, or $q = 2^m$

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- Inversion: $a \in \mathbb{F}_p,\ a \neq 0 \Rightarrow \exists\ a^{-1} \in \mathbb{F}_p : a \cdot a^{-1} = 1$

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Remark: We are interested in **two kinds** of bases: **polynomial** bases and **normal** bases

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Identities: $(1) = (00...01)$ $(0) = (00...00)$

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$$\mathbb{F}_{2^m} = \{a = \sum_{i=0}^{m-1} a_i \beta^{2^i} : a_i \in \{0, 1\}\}$$

$$\mathbb{F}_{2^m} = \{(a_0 a_1 ... a_{m-1}) : a_i \in \{0, 1\}\}$$

Identities: $(1) = (11...11)$ $(0) = (00...00)$

Field operations

- Addition: $a = (a_0 a_1 ... a_{m-1}), b = (b_0 b_1 ... b_{m-1}) \in \mathbb{F}_{2^m}$
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- Multiplication: with use of **Gaussian normal basis (GNB)**

- Inversion: $a \in \mathbb{F}_{2^m},\ a \neq 0 \Rightarrow \exists\ !\ a^{-1} \in \mathbb{F}_{2^m}\colon\ a \cdot a^{-1} = 1$

Elliptic curves over $\mathbb{F}_p$

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Definition

Let $p > 3$ be a **prime** number, $a, b \in \mathbb{F}_p$: $4a^3 + 27b^2 \neq 0 \mod p$
$E(\mathbb{F}_p) = \{(x, y) \in \mathbb{F}_p \times \mathbb{F}_p : \; y^2 = x^3 + ax + b \;\} \cup$
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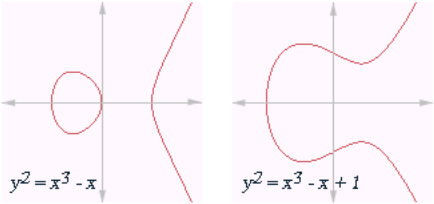

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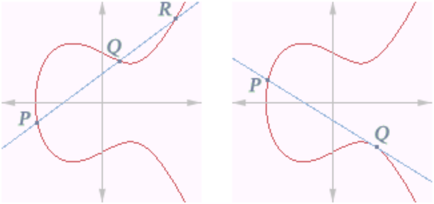

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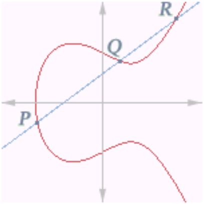

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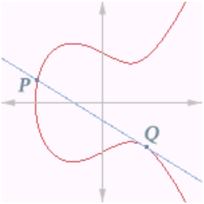

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Remark: Together with this addition operation a set $E(\mathbb{F}_p)$ forms a **group** with $\mathcal{O}$ serving as **identity**

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$\exists$ a **generator** $G \in E(\mathbb{F}_q)$: $E(\mathbb{F}_q) = \{kG : 0 \leq k \leq n - 1\}$

ECDLP

$\forall G \in E(\mathbb{F}_q)$ of prime order n generates a **cyclic subgroup**

$$(\mathcal{O}, G, 2G, 3G, ..., (n-1)G) \;\Leftrightarrow\; (e, g, g^2, g^3, ..., g^{(n-1)})$$

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Definition

Elliptic curve discrete logarithm problem (ECDLP):
Find k for given points G and kG, where $0 \leq k \leq n-1$

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- **Other methods**:
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- $\Rightarrow Q$ is **valid**, otherwise - **invalid**

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If a signature (r, s) on a message m was **indeed** generated

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$\Rightarrow k \equiv (e + dr)s^{-1} \equiv (e + dr)w \equiv u_1 + du_2 \bmod n$

$\Rightarrow u_1 G + u_2 Q = (u_1 + du_2)G = kG \;\Rightarrow\; r = v$

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- **The construction of r**:

  $r = (g^k \mod p) \mod q \longrightarrow$

  $r = x_1 \mod n$, where $(x_1, y_1) = kG$

Not included

- Security
- Known attacks
- Implementation
- Interoperability
- ECDSA standarts
- Recommended elliptic curves

Thank you for your attention!

D. Johnson, A. Menezes, S. Vanstone: The Elliptic Curve Digital Signature Algorithm (ECDSA). 2001