JASS '07 - Course 1

Basic Concepts of Differential Algebra

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Abstract

Modern computer algebra systems symbolically integrate a vast variety of functions. To reveal the underlying structure it is necessary to understand infinite integration not only as an analytical problem but as an algebraic one. Introducing the differential field of elementary functions we sketch the mathematical tools like Liouville's Principle used in modern algorithms. We present Hermite's method for integration of rational functions as well as the Rothstein/Trager method for rational and for elementary functions. Further applications of the mentioned algorithms in the field of ODE's conclude this paper.

Contents

1 Introduction

The problem of indefinite integration is one of the easiest problems of mathematics to describe: given a function $f(x)$, find $g(x)$ such that

$$g'(x) = f(x).$$

If such a function can be found then one writes

$$\int f(x)\,\mathrm{dx} = g(x).$$

As simple as this problem statement is as hard is its solution. Only for a limited class of functions does there even exist a solution in a closed form. We will introduce some basic concepts to describe a function classes for which the question "'Does f have an indefinite integral within this class?"' is decidable and give algorithms to construct the indefinite integral where possible.

2 Basics

2.1 Differential Fields and Ideals

Definition 2.1 (Differential Field)
A differential field is a field F of characteristic 0 on which is defined a mapping $D : F \to F$ satisfying, for all $f, g \in F$:

$$D(f+g) = D(f) + D(g), \tag{1}$$
$$D(f \cdot g) = f \cdot D(g) + g \cdot D(f). \tag{2}$$

The mapping D is called a derivation or differential operator.

A differential operator D has the following expected properties:

Lemma 2.2
If D is a differential operator on a differential field F then the following properties hold:

i. $D(0) = D(1) = 0$;

ii. $D(-f) = -D(f)$, *for all* $f \in F$;

iii. $D\left(\frac{f}{g}\right) = \frac{g \cdot D(f) - f \cdot D(g)}{g^2}$, *for all* $f, g \in F$ $(g \neq 0)$;

iv. $D(f^n) = nf^{n-1}D(f)$, *for all* $n \in \mathbb{Z}, f \in F$ $(f \neq 0)$.

v. $\int f \cdot D(g) = f \cdot g - \int D(f) \cdot g$. *(integration by parts)*

(without proof)

Definition 2.3 (Differential Extension Field)
Let F and G be differential fields with differential operators D_F and D_G, respectively. Then G is a differential extension field of F if G is an extension field of F and

$$D_F(f) = D_G(f) \text{ for all } f \in F.$$

Definition 2.4 (Field of Constants)
Let F be a differential field with differential operator D. The field of constants (or constant field) of F is the subfield of F defined by

$$K = \{c \in F : D(c) = 0\}.$$

Lemma 2.5
For the differential field $\mathbb{Q}(x)$ with differential operator D satisfying $D(x) = 1$, the constant field is $\mathbb{Q}$.

Proof
If $c \in \mathbb{Q}$ then $D(c) = 0$. Conversely, suppose $D(r) = 0$ for $r \in Q(x)$. We must prove $r \in \mathbb{Q}$. Write $r = \frac{p}{q}$ for $p, q \in \mathbb{Q}[x]$, with $q \neq 0$, $\gcd(p, q) = 1$. We have

$$D(r) = \frac{q \cdot D(p) - p \cdot D(q)}{q^2}$$

giving

$$D(p) = \frac{p \cdot D(q)}{q} \in \mathbb{Q}[x]$$

Since p and q have no common factors it follows

$$q \mid D(q)$$

As $\deg(q) > \deg(D(q))$ for all $q \in \mathbb{Q}[x] \setminus \mathbb{Q}$ it follows

$$D(q) = 0.$$

And the previous relationship yields $D(p) = 0$. We have thus proved that $p, q \in \mathbb{Q}$ as desired. □

Lemma 2.6
For the rational function $1/x \in Q(x)$, there does not exist a rational function $r \in \mathbb{Q}(x)$ such that $D(r) = 1/x$.

Proof
Assume $r = p/q \in \mathbb{Q}(x)$ satisfies $D(p/q) = 1/x$, with $p, q \in \mathbb{Q}[x]$ and $\gcd(p, q) = 1$. Then

$$\frac{q \cdot D(p) - p \cdot D(q)}{q^2} = \frac{1}{x}$$

so

$$x \cdot q \cdot D(p) - x \cdot p \cdot D(q) = q^2 \tag{3}$$

By equation (3) $x \mid q$. So we write $q = x^n \cdot \hat{q}$, where $\hat{q} \in \mathbb{Q}[x]$ and $\gcd(x, \hat{q}) = 1$. Substituting we get

$$x^{n+1} \cdot \hat{q} \cdot D(p) - nx^n \cdot p \cdot \hat{q} - x^{n+1} \cdot p \cdot D(\hat{q}) = x^{2n} \cdot \hat{q}^2$$

which simplifies to

$$x \cdot \left(\hat{q} \cdot D(p) - p \cdot D(\hat{q}) - x^{n-1} \cdot \hat{q}^2\right) = n \cdot p \cdot \hat{q}$$

Since $\gcd(\hat{q}, x) = 1$, we must have that $x \mid p$. But then p and q have a common factor, a contradiction. $\square$

As we have seen there exist $f \in \mathbb{Q}(x)$ such that $\int f = g$ has no solution in $\mathbb{Q}(x)$. Nevertheless there exists a indefinite integral for every $f \in \mathbb{Q}(x)$. We have to extend the differential field F in order to find the solution g satisfying $\int f = g$.

Definition 2.7 (Logarithmic Functions)
Let F be a differential field and let G be a differential extension field of F. If, for a given $\theta \in G$, there exists an element $u \in F$ such that

$$D(\theta) = \frac{D(u)}{u}$$

then θ is called logarithmic over F and we write $\theta = \log(u)$.

2.2 Rational Part of the Integral: Hermite's Method

Let K be a field and $f, g \in K[x]$ be nonzero and relatively prime, and suppose we want to compute $\int f/g$. The idea is to find first $a, b, c, d \in K[x]$ with

$$\int \frac{f}{g} = \frac{c}{d} + \int \frac{a}{b},$$

$\deg a < \deg b$ and b is monic and square-free. (Recall that b is square-free iff $\gcd(b, b') = 1$.) The rational function c/d is called the rational part, $\int a/b$ the logarithmic part of the integral; we deal with the latter in the next section.
Hermite's method proceeds as follows. Let $p/q \in K(x)$ be normalized such that $\gcd(p, q) = 1$ and q is monic. Apply Euclidean division to p and q yielding polynomials $s, r \in K[x]$ such that $p = q \cdot s + r$ with $r = 0$ or $\deg(r) < \deg(q)$. We then have

$$\int \frac{p}{q} = \int s + \int \frac{r}{q}.$$

Integrating the polynomial s is trivial. To integrate the proper fraction r/q, compute the square-free factorization of the denominator

$$q = \prod_{i=1}^{k} q_i^i$$

where each q_i ($1 \le i \le k$) is monic and square-free, $\gcd(q_i, q_j) = 1$ for $i \ne j$, and $\deg(q_k) > 0$. Compute the partial fraction expansion of the integrand $r/q \in K(x)$ in the form

$$\frac{r}{q} = \sum_{i=1}^{k} \sum_{j=1}^{i} \frac{r_{ij}}{q_i^j}$$

where $r_{ij} \in K[x]$ and

$$\deg(r_{ij}) < \deg(q_i) \text{ if } \deg(q_i) > 0,\ r_{ij} = 0 \text{ if } q_i = 1.$$

The integral of r/q can then be expressed in the form

$$\int \frac{r}{q} = \sum_{i=1}^{k} \sum_{j=1}^{i} \int \frac{r_{ij}}{q_i^j}.$$

Now we apply reductions on the integrals of the right hand side until each integral that remains has a square-free denominator. The main tools in this process will be integration by parts and application of the extended Euclidean algorithm.
Consider a particular nonzero integrand r_{ij}/q_i^j with $j > 1$. Since q_i is square-free, $\gcd(q_i, q_i') = 1$ so we may apply the extended Euclidean algorithm to compute polynomials $s, t \in K[x]$ such that

$$s \cdot q_i + t \cdot q_i' = r_{ij}$$

where $\deg(s) < \deg(q_i) - 1$ and $\deg(t) < \deg(q_i)$. Dividing by q_i^j yields

$$\int \frac{r_{ij}}{q_i^j} = \int \frac{s}{q_i^{j-1}} + \int \frac{tq_i'}{q_i^j}.$$

Applying integration by parts we get

$$\int \frac{r_{ij}}{q_i^j} = \frac{-t/(j-1)}{q_i^{j-1}} + \int \frac{s + t'/(j-1)}{q_i^{j-1}}.$$

If $j-1 > 1$ then this reduction process may be applied again. Otherwise, if $j-1 = 1$ then this integral contributes to the logarithmic part to be considered in the next subsection. Also note that the numerator of the new integrand satisfies the degree constraint

$$\deg(s + t'/(j-1)) \leq \max\{\deg(s), \deg(t')\} < \deg(q_i) - 1.$$

Example 1
Consider $\int f$ where $f \in \mathbb{Q}(x)$ is

$$f = \frac{x^4 + x^3 + 2x^2 + 2x + 1}{x^3 + 2x^2 + x}.$$

Euclidean division yields

$$f = P + \frac{p}{q}$$

where $P = x - 1$, $p = 3x^2 - 3x + 1$ and $q = x^3 + 2x^2 + x$. The square-free factorization of q is

$$q = x \cdot (x+1)^2.$$

And the partial fraction expansion yields

$$\frac{r}{q} = \frac{1}{x} + \frac{2x+1}{(x+1)^2}.$$

Thus $\int f$ simplifies to

$$\int f = \int P + \int \frac{1}{x} + \int \frac{2x+1}{(x+1)^2}.$$

We apply one reduction step to the last summand:

$$\int \frac{2x+1}{(x+1)^2} = \frac{1}{x+1} - \int \frac{2}{x+1}$$

which gives

$$\begin{aligned}\int f &= \int P + \int \frac{1}{x} + \left(\frac{1}{x+1} - \int \frac{2}{x+1}\right)\\ &= x^2 + x + \frac{1}{x+1} + \int \frac{1}{x} + (-2)\cdot \int \frac{1}{x+1}\end{aligned}$$

As shown in 2.6 there is no $g \in \mathbb{Q}(x)$ such that $g' = 1/x$. Thus this integral can not be further simplified in $\mathbb{Q}(x)$.

Example 2
Consider $\int f$ where $f \in \mathbb{Q}(x)$ is

$$f = \frac{1}{x^2+1}.$$

In this case $q = x^2 + 1$ does not split into linear factors. While in the last example $g = \log x$ is a well known solution to $g' = 1/x$, things are more difficult in this case. It is necessary to extend the constant field $\mathbb{Q}$ to split q into linear factors and find simple logarithmic integrals:

$$\frac{1}{x^2+1} = \frac{-1/2i}{x-i} + \frac{1/2i}{x+i}.$$

Thus

$$\int \frac{1}{x^2+1} = -\frac{1}{2}i \cdot \log(x-i) + \frac{1}{2}i \cdot \log(x+i)$$

which clearly is not in $\mathbb{Q}(x)$.

2.3 Logarithmic Part of the Integral: The Rothstein/Trager Method

Consider now the problem of expressing the logarithmic part of the integral of a rational function, which is an integral of the form

$$\int \frac{a}{b}$$

where $a, b \in K[x]$, $\deg(a) < \deg(b)$, and b is monic and square-free. As noted in Example 2 it may be necessary to extend the constant field K to $K(\alpha_1, \ldots, \alpha_k)$ where α_i $(1 \leq i \leq k)$ are algebraic numbers over K. However we would like to express the integral using the minimal algebraic extension field.
Ignoring any concern about the number of algebraic extensions, let the denominator $b \in K[x]$ be completely factored over its splitting field K_b into the form

$$b = \prod_{i=1}^{m} (x - \beta_i)$$

where β_i $(1 \leq i \leq m)$ are m distinct elements of K_b, an algebraic extension of K. Then the integrand can be expressed in a partial fraction expansion of the form

$$\frac{a}{b} = \sum_{i=1}^{m} \frac{\gamma_i}{x - \beta_i} \text{ where } \gamma_i, \beta_i \in K_b$$

and so

$$\int \frac{a}{b} = \sum_{i=1}^{m} \gamma_i \cdot \log(x - \beta_i)$$

with the result of the integration expressed in the extension field
$K_b(x, \log(x - \beta_1), \ldots, \log(x - \beta_m))$.
When K is a field which is not algebraically closed, such as $\mathbb{Q}$, then the above method has serious practical difficulties. In the worst case, the splitting field of a degree-m polynomial is of degree $m!$ over K. The following algorithm computes the integral of $f \in K(x)$ using the minimal extension of the differential field $K(x)$:

Theorem 1 (Rothstein/Trager Method - Rational Function Case)
Let $K^*(x)$ be a differential field over some constant field K^*. Let $a, b \in K^*[x]$ be such that $\gcd(a, b) = 1$, *with b monic and square-free, and* $\deg(a) < \deg(b)$. *Suppose that*

$$\int \frac{a}{b} = \sum_{i=1}^{n} c_i \cdot \log(v_i)$$

where $c_i \in K^*$ $(1 \leq i \leq n)$ are distinct nonzero constants and $v_i \in K^*[x]$ $(1 \leq i \leq n)$ are monic, square-free, pairwise relatively prime polynomials of positive degree. Then c_i $(1 \leq i \leq n)$ are the distinct roots of the polynomial

$$R(z) = \operatorname{res}_x(a - zb', b) \in K^*[x]$$

and v_i $(1 \leq i \leq n)$ are the polynomials

$$v_i = \gcd(a - c_i b', b) \in K^*[x].$$

For the detailed proof we refer the reader to [Ged92], pp.495-497.

The following theorem states that the method of Rothstein and Trager is completely general and leads to a minimal constant field:

Theorem 2
Let $K(x)$ be a differential field over a constant field K. Let $a, b \in K[x]$ be such that $\gcd(a, b) = 1$, *b monic and square-free, and* $\deg(a) < \deg(b)$. *Let K^* be the minimal algebraic extension field of K such that the integral can be expressed in the form*

$$\int \frac{a}{b} = \sum_{i=1}^{n^*} c_i^* \cdot \log(v_i^*)$$

where $c_i^* \in K^*, v_i^* \in K^*[x]$. Then

$$K^* = K(c_1, \ldots, c_n)$$

where c_i $(1 \leq i \leq n)$ are the distinct roots of the polynomial

$$R(z) = \operatorname{res}_x(a - zb', b) \in K[z].$$

In other words, K^* is the splitting field of $R(z) \in K[z]$. Moreover the formulas in Theorem 1 may be used to calculate the integral using the minimal constant field.

For the proof see [Ged92], pp.497-498.

Example 3
Let us apply the Rothstein/Trager method to compute the following integral

$$\frac{a}{b} = \frac{1}{x^3+x} \in \mathbb{Q}(x).$$

Since b is square-free and $\deg(a) < \deg(b)$, the integral has only a logarithmic part. First compute the resultant

$$\begin{aligned} R(z) &= \mathrm{res}_x(a - z \cdot b', b) = \mathrm{res}_x(-3zx^2 + (1-z), x^3 + x) \\ &= \det \begin{pmatrix} -3z & 0 & 1-z & & \\ & -3z & 0 & 1-z & \\ & & -3z & 0 & 1-z \\ 1 & 0 & 1 & 0 & \\ & 1 & 0 & 1 & \end{pmatrix} = 4z^3 + 3z + 1 \in \mathbb{Q}[z] \end{aligned}$$

Next computing the complete factorization of $R(z)$ in the domain $\mathbb{Q}(z)$ gives

$$R(z) = -4(z-1)(z+\frac{1}{2})^2.$$

In this case $R(z)$ completely splits over the constant field $\mathbb{Q}$ and therefore no algebraic number extensions are required to express the integral. The distinct roots of $R(z)$ are

$$c_1 = 1, c_2 = -1/2.$$

The corresponding log arguments:

$$\begin{aligned} v_1 &= \gcd(a - c_1 \cdot b', b) = x \in \mathbb{Q}[x] \\ v_2 &= \gcd(a - c_2 \cdot b', b) = x^2 + 1 \in \mathbb{Q}[x] \end{aligned}$$

Hence,

$$\begin{aligned} \int \frac{1}{x^3+x} &= c_1 \cdot \log(v_1) + c_2 \cdot \log(v_2) \\ &= \log(x) - 1/2 \log(x^2+1) \in \mathbb{Q}(x, \log(x), \log(x^2+1)). \end{aligned}$$

If $R(z)$ does not split over the constant field it is necessary to adjoin algebraic numbers to the constant field

Example 4
The integral $\int a/b$ with

$$\frac{a}{b} = \frac{1}{x^2-2} \in \mathbb{Q}(x)$$

has

$$R(z) = -8(z^2 - 1/8) \in \mathbb{Q}[z]$$

and thus the splitting field is $\mathbb{Q}(\sqrt{2})$ yielding the final result

$$\int \frac{1}{x^2-2} = \frac{1}{4} \cdot \sqrt{2} \cdot \log(x - \sqrt{2}) - \frac{1}{4} \cdot \sqrt{2} \cdot \log(x + \sqrt{2})$$

with the answer in the extension field $\mathbb{Q}(\sqrt{2})(x, \log(x-\sqrt{2}), \log(x+\sqrt{2}))$.

3 The Risch Integration Algorithm

3.1 Elementary Functions

Before studying more powerful integration algorithms we define a class of functions for which the problem of the existence of an indefinite integral is decidable. This class of functions will be a differential field commonly referred to as the *elementary functions*. Those include rational functions (as studied in Section 2.2ff.), exponentials, logarithms, algebraic functions (e.g. nth roots), as well as trigonometric, inverse trigonometric, hyperbolic, and inverse hyperbolic functions. Any finite composition of the above functions is again an elementary function. Before we continue with the integration algorithms we introduce a basic framework to simplify notation. This is necessary to reveal the underlying structure as the following example shows:

$$\int \cos(x) = \sin(x);$$
$$\int \frac{1}{\sqrt{1-x^2}} = \arcsin(x);$$
$$\int \operatorname{arccosh}(x) = x \operatorname{arccosh}(x) - \sqrt{x^2-1}.$$

In these examples, there does not appear to be a regular relationship between the integrand and the resulting integral. As we express these integrals in a new form, we see that functions appearing in the integrand generally appear also in the expression for the integral, plus new logarithmic extensions may appear:

$$\int \left(\frac{1}{2}\exp(ix) + \frac{1}{2}\exp(-ix)\right) = -\frac{1}{2}i\exp(ix) + \frac{1}{2}i\exp(-ix);$$
$$\int \frac{1}{\sqrt{1-x^2}} = -i\log(\sqrt{1-x^2}+ix);$$
$$\int \log(x+\sqrt{x^2-1}) = x\log(x+\sqrt{x^2-1}) - \sqrt{x^2-1}.$$

This more structured approach will help us device efficient methods to find integrals where they exist. First we introduce some notation:

Definition 3.1
Let F be a differential field and let G be a differential extension field of F.

i. For an element $\theta \in G$, if there exists an element $u \in F$ such that

$$\theta' = \frac{u'}{u}$$

then θ is called logarithmic over F and we write $\theta = \log(u)$.

ii. For an element $\theta \in G$, if there exists an element $u \in F$ such that

$$\frac{\theta'}{\theta} = u'$$

then θ is called exponential over F and we write $\theta = \exp(u)$.

iii. For an element $\theta \in G$, if there exists a polynomial $p \in F[z]$ such that

$$p(\theta) = 0$$

then θ is called algebraic over F.

Definition 3.2
Let F be a field and let G be an extension field of F. An element $\theta \in G$ is called transcendental over F if θ is not algebraic over F.

Definition 3.3
Let F be a differential field and let G be a differential extension field of F. G is called a transcendental elementary extension of F if it is of the form

$$G = F(\theta_1, \dots, \theta_n)$$

where for each $i = 1, \dots, n$, θ_i is transcendental and either logarithmic or exponential over the field $F_{i-1} = F(\theta_1, \dots, \theta_{i-1})$. G is called an elementary extension of F if it is of the form

$$G = F(\theta_1, \dots, \theta_n)$$

where for each $i = 1, \dots, n$, θ_i is either logarithmic, or exponential, or algebraic over the field $F_{i-1} = F(\theta_1, \dots, \theta_{i-1})$. (In this notation, $F_0 = F$.)

Definition 3.4 (Transcendental Elementary Functions)
Let $K(x)$ be a differential field of rational functions over a constant field K which is a subfield of the field of complex numbers. If F is a transcendental elementary extension of $K(x)$ then F is called a field of transcendental elementary functions. Similarly, if F is an elementary extension of $K(x)$ then F is called a field of elementary functions.

Definition 3.5
An element θ is monomial over a differential field F if

i. $F(\theta)$ and F have the same constant field,

ii. θ is transcendental over F,

iii. θ is either exponential or logarithmic over F.

As we have defined the basic terms we can now proceed to one of the core-theorems for the integration of elementary functions.

3.2 Liouville's Principle

In Section 2.2 we have shown that rational functions always have an integral which can be expressed as a transcendental elementary function. The fundamental result on elementary function integration was first presented by Liouville in 1833. It is the basis of the algorithmic approach to elementary function integration.

Theorem 3 (Liouville's Principle)
Let F be a differential field with constant field K. For $f \in F$ suppose that the equation $g' = f$ has a solution $g \in G$ where G is an elementary extension of F having the same constant field K. Then there exist $v_0, v_1, \dots, v_m \in F$ and constants $c_1, \dots, c_m \in K$ such that

$$f = v_0' + \sum_{i=1}^{m} c_i \frac{v_i'}{v_i}.$$

In other words, such that

$$\int f = v_0 + \sum_{i=1}^{m} c_i \log(v_i).$$

For an elegant proof of Theorem 3 see [Ros72].

3.3 The Risch Algorithm

In this section we develop an effective decision procedure for the elementary integration of any function which belongs to a field of transcendental elementary functions (see Definition 3.4). The decision procedure will determine $\int f$ if it exists as an elementary function. Otherwise, it constructs a proof of the nonexistence of an elementary integral.

Given an integrand f, the first step is to determine a description $K(x, \theta_1, \ldots, \theta_n)$ of a field of transcendental elementary functions in which f lies (if f lies in such a field). To handle this step in our integration algorithm, we will first convert all trigonometric (or related) functions into their exponential and logarithmic forms. Then the algebraic relationships which exist among the various exponential and logarithmic functions are determined. For the remainder of this section, we well assume that a purely transcendental description $K(x, \theta_1, \ldots, \theta_n)$ has been given for the integrand f. Since each extension θ_i is a transcendental symbol, the integrand may be manipulated as a rational function in these symbols. The integration algorithm for transcendental functions will follow steps that are very reminiscent of the development in the previous section of the rational function integration algorithm, in particular Hermite's method and the Rothstein/Trager method. Given an integrand $f \in K(x, \theta_1, \ldots, \theta_n)$, it may be viewed as a rational function in the last extension $\theta = \theta_n$

$$f(\theta) = \frac{p(\theta)}{q(\theta)} \in F_{n-1}(\theta)$$

where $F_{n-1} = K(x, \theta_1, \ldots, \theta_{n-1})$. We may assume that $f(\theta)$ is normalized such that $p(\theta), q(\theta) \in F_{n-1}[\theta]$ satisfy $\gcd(p(\theta), q(\theta)) = 1$ and that $q(\theta)$ is monic. Keep in mind that $\int f(\theta)$ is integration with respect to x. We will continue to use $'$ for differentiation with respect to x only, using $\frac{d}{d\theta}$ for differentiation with respect to θ. The algorithm is recursive, so that when treating $\int f(\theta)$ there will be recursive invocations to integrate functions in the field F_{n-1}. The base of the recursion is integration in the field $F_0 = K(x)$ which is handled by Hermite's method or the method by Rothstein/Trager.

The Risch Algorithm for Logarithmic Extensions

Consider first the case where θ is logarithmic, with say $\theta' = u'/u$ and $u \in F_{n-1}$. Proceeding as in Hermite's method, apply Euclidean division to $p(\theta), q(\theta) \in F_{n-1}[\theta]$ yielding polynomials $s(\theta), r(\theta) \in F_{n-1}[\theta]$ such that

$$p(\theta) = q(\theta) \cdot s(\theta) + r(\theta) \text{ with } r(\theta) = 0 \text{ or } \deg(r(\theta)) < \deg(q(\theta)).$$

We then have

$$\int f(\theta) = \int s(\theta) + \int \frac{r(\theta)}{q(\theta)}.$$

We refer to the first integral on the right hand side of this equation as the integral of the *polynomial part* of $f(\theta)$, and to the second integral as the integral of the *rational part* of $f(\theta)$. Unlike the case of pure rational function integration, the integration of the polynomial part is not trivial.

We first concentrate on the rational part of the integral. The following theorem guarantees that if $\int a(\theta)/b(\theta)$ is elementary then it can be expressed in the form

$$\int \frac{a(\theta)}{b(\theta)} = \sum_{i=1}^{m} c_i \log(v_i(\theta)),$$

gives an efficient method to determine when this form exists (by looking at the primitive part of the resultant $R(z)$), gives an efficient method to compute the factors $v_i(\theta)$ by

$$v_i(\theta) = \gcd(a(\theta) - c_i \cdot b(\theta)', b(\theta)) \in F_{n-1}(c_1, \ldots, c_m)[\theta],$$

and guarantees that the result is expressed using the minimal algebraic extension field.

Theorem 4 (Rothstein/Trager Method - Logarithmic Case)
Let F be a field of elementary functions with constant field K. Let θ be transcendental and logarithmic over F (i.e. $\theta' = u'/u$ for some $u \in F$) and suppose that the transcendental elementary extension $F(\theta)$ has the same constant field K. Let $a(\theta)/b(\theta) \in F(\theta)$ where $a(\theta), b(\theta) \in F[\theta]$, $\gcd(a(\theta), b(\theta)) = 1$, $\deg(a(\theta)) < \deg(b(\theta))$, and $b(\theta)$ is monic and square-free.

i. $\int \frac{a(\theta)}{b(\theta)}$ *is elementary if and only if all the roots of the polynomial*

$$R(z) = \mathrm{res}_\theta(a(\theta) - z \cdot b(\theta)', b(\theta)) \in F[z]$$

are constants.

ii. If $\int \frac{a(\theta)}{b(\theta)}$ *is elementary then*

$$\frac{a(\theta)}{b(\theta)} = \sum_{i=1}^{m} c_i \frac{v_i(\theta)'}{v_i(\theta)} \tag{4}$$

where c_i $(1 \le i \le m)$ are the distinct roots of $R(z)$ and $v_i(\theta)$ $(1 \le i \le m)$ are defined by

$$v_i(\theta) = \gcd(a(\theta) - c_i \cdot b(\theta)', b(\theta)) \in F(c_1, \ldots, c_m)[\theta].$$

iii. Let F^* be the minimal algebraic extension field of F such that $a(\theta)/b(\theta)$ can be expressed in the form (4) with constant $c_i \in F^*$ and with $v_i(\theta) \in F^*[\theta]$. Then $F^* = F(c_1, \ldots, c_m)$ where c_i $(1 \le i \le m)$ are the distinct roots of $R(z)$.

For the proof see [Ged92] pp. 538-540.

Theorem 4 gives the integral for the rational part of an elementary function if there exists one. Now we consider the polynomial part in a logarithmic extension. Unlike the integration of the polynomial part of rational functions the integration of the polynomial part of elementary function is not trivial. Since the polynomial part is a polynomial $p(\theta) \in F_{n-1}[\theta]$ and θ is logarithmic over F_{n-1} there is no simple integration formula. (Keep in mind that we are integrating with respect to x.) If $\int p(\theta)$ is elementary then

$$p(\theta) = v_0(\theta)' + \sum_{i=1}^{m} c_i \frac{v_i(\theta)'}{v_i(\theta)}$$

by Liouville's Principle. Using this equality one gets a system of equations during the integration process which is applied recursively. We omit further details.

Example 5
The integral

$$\int \frac{1}{\log(x)}$$

has integrand

$$f(\theta) = \frac{1}{\theta} \in \mathbb{Q}(x, \theta)$$

where $\theta = \log(x)$. Applying the Rothstein/Trager method, we compute

$$R(z) = \mathrm{res}_\theta\left(1 - \frac{z}{x}, \theta\right) = 1 - \frac{z}{x} \in \mathbb{Q}(x)[z].$$

Since $R(z)$ has a non constant root, we conclude that the integral is not elementary.

Example 6
The integral

$$\int \frac{1}{x \log(x)}$$

has integrand

$$f(\theta) = \frac{1/x}{\theta} \in \mathbb{Q}(x, \theta)$$

where $\theta = \log(x)$. Applying the Rothstein/Trager method, we compute

$$R(z) = \mathrm{res}_\theta\left(\frac{1}{x} - \frac{z}{x}, \theta\right) = \frac{1}{x} - \frac{z}{x} \in \mathbb{Q}(x)[z].$$

Since $R(z)$ has the constant root 1, the integral is elementary. Specifically,

$$\begin{aligned} c_1 &= 1, \\ v_1(\theta) &= \gcd(1/x - \theta', \theta) = \gcd(1/x - 1/x, \theta) = \theta, \end{aligned}$$

and

$$\int \frac{1}{x \log(x)} = c_1 \log(v_1(\theta)) = \log(\log(x)).$$

The Risch Algorithm for Exponential Extensions

Suppose that the last extension θ is exponential, specifically that $\theta'/\theta = u'$ where $u \in F_{n-1}$. Again we want to compute $\int f(\theta)$ with

$$f(\theta) = \frac{p(\theta)}{q(\theta)} \in F_{n-1}[\theta]$$

where $p(\theta), q(\theta) \in F_{n-1}[\theta], \gcd(p(\theta), q(\theta)) = 1$, and $q(\theta)$ monic. As in the previous case we divide f into a polynomial and a rational part. We first handle the rational part:

Theorem 5 (Rothstein/Trager Method - Exponential Case)
Let F be a field of elementary functions with constant field K. Let θ be transcendental and exponential over F (i.e. $\theta'/\theta = u'$ for some $u \in F$) and suppose that the transcendental elementary extension $F(\theta)$ has the same constant field K. Let $a(\theta)/b(\theta) \in F(\theta)$ where $a(\theta), b(\theta) \in F[\theta]$, $\gcd(a(\theta), b(\theta)) = 1$, $\deg(a(\theta)) < \deg(b(\theta))$, $\theta \not| b(\theta)$, and with $b(\theta)$ monic and square-free.

i. $\int \frac{a(\theta)}{b(\theta)}$ is elementary if and only if all the roots of the polynomial

$$R(z) = \mathrm{res}_\theta(a(\theta) - z \cdot b(\theta)', b(\theta)) \in F[z]$$

are constants.

ii. *If* $\int \frac{a(\theta)}{b(\theta)}$ *is elementary then*

$$\frac{a(\theta)}{b(\theta)} = g' + \sum_{i=1}^{m} c_i \frac{v_i(\theta)'}{v_i(\theta)} \tag{5}$$

where c_i $(1 \leq i \leq m)$ *are the distinct roots of* $R(z)$ *and* $v_i(\theta)$ $(1 \leq i \leq m)$ *are defined by*

$$v_i(\theta) = \gcd(a(\theta) - c_i \cdot b(\theta)', b(\theta)) \in F(c_1, \ldots, c_m)[\theta],$$

and where $g \in F(c_1, \ldots, c_m)$ *is defined by*

$$g' = -\left(\sum_{i=1}^{m} c_i \deg(v_i(\theta))\right) u'.$$

iii. *Let* F^* *be the minimal algebraic extension field of* F *such that* $a(\theta)/b(\theta)$ *can be expressed in the form (5) with constant* $c_i \in F^*$ *and with* $v_i(\theta) \in F^*[\theta]$. *Then* $F^* = F(c_1, \ldots, c_m)$ *where* c_i $(1 \leq i \leq m)$ *are the distinct roots of* $R(z)$.

For the proof see [Ged92] pp. 555-557.

Example 7
Consider the integral

$$\int \frac{1}{\exp(x) + 1}.$$

this has integrand

$$f(\theta) = \frac{1}{\theta + 1} \in \mathbb{Q}(x, \theta)$$

where $\theta = \exp(x)$. Applying Rothstein/Trager method, we compute

$$R(z) = \mathrm{res}_\theta(1 - z\theta, \theta + 1) = -z - 1 \in \mathbb{Q}(x)[z].$$

Since $R(z)$ has the constant root -1, the integral is elementary. Specifically,

$$c_1 = -1, v_1(\theta) = \gcd(1 + \theta, \theta + 1) = \theta + 1,$$

and

$$\int \frac{1}{\exp(x) + 1} = -c_1 \deg(v_1(\theta))x + c_1 \log(v_1(\theta)) = x - \log(\exp(x) + 1).$$

Example 8
The integral

$$\int \frac{x}{\exp(x) + 1}.$$

has integrand

$$f(\theta) = \frac{x}{\theta + 1} \in \mathbb{Q}(x, \theta)$$

where $\theta = \exp(x)$. Applying Rothstein/Trager method, we compute

$$R(z) = \mathrm{res}_\theta(x - z\theta, \theta + 1) = -z - x \in \mathbb{Q}(x)[z].$$

Thus the integral is not elementary as $R(z)$ has the root x.

Integration of Algebraic Functions

The nontrivial integration of the polynomial part of the integral involves Liouville's Principle as well as solving the resulting system of equations. Due to the special structure of exponential extensions these resulting equations are differential equations known as *Risch differential equations*. They are of the form

$$y' + fy = g$$

where the given functions are $f, g \in F_{n-1}$ and we must determine a solution $y \in \overline{F}_{n-1}$. It might seem that solving these equations is a harder problem than the original integration problem. But since the solution to the differential equation is restricted to lie in the same field as the functions f and g, it is possible to solve the Risch differential equation or else to prove that there is no solution of the desired form.

It remains to take care of the integration of algebraic functions. Trager presented an approach in his PhD thesis [Tra84] relying on algebraic geometry. Take the problem of integrating

$$\int \frac{p(x,y)}{q(x,y)} \,\mathrm{dx}$$

with y algebraic over the function field $K(x)$ with K the field of constants. As y is algebraic over $K(x)$ there is $F(x,y) \in K(x)[y]$, $F(x,y)$ irreducible satisfying

$$F(x,y) = 0.$$

This highlights the core-problem of symbolic integration for algebraic extensions. Because of the logarithmic terms which we know exist in the integral (by Liouville's Principle), we wish to have knowledge of the poles of the integrand. In the transcendental case, such information was straightforward: the poles of a reduced rational function are the zeros of the denominator and vice versa. We do not have the same property here. For example, if

$$F(x,y) = y^4 - x^3$$

then we represent $f = x^2/y^2$ as

$$f = \frac{x^2}{y^2} = \frac{y^2}{x}.$$

As such, f appears to have one pole of order 1 at the point $x = 0$. However, we have

$$f^2 = \frac{y^4}{x^2} = \frac{x^3}{x^2} = x.$$

Therefore f^2 (and hence also f) has no poles. Resolving this obstacle involves algorithms found in [Tra84] or [Bra88].

4 Applications

4.1 Systems of Linear Differential Equations

In the previous section we gave a solution to a special differential equation, the integration problem

$$g = f'.$$

This problem has been studied extensively. Now we turn to a more general class of differential equations: upper triangular systems of first-order linear equations. Those are defined as follows:

Definition 4.1 (System of Linear Differential Equations)
Let f_i $(1 \leq i \leq n)$ be functions whose first partial derivatives are continuous. A system of first-order differential equations is written in the form

$$\begin{cases} \frac{\partial x_1}{\partial t} = f_1(t, x_1, \dots, x_n) \\ \frac{\partial x_2}{\partial t} = f_2(t, x_1, \dots, x_n) \\ \qquad \vdots \\ \frac{\partial x_n}{\partial t} = f_n(t, x_1, \dots, x_n) \end{cases} \tag{6}$$

Where

$$f_i(t, x_1, \dots, x_n) = p_{i1}(t)x_1 + \cdots + p_{in}(t)x_n + g_i(t)$$

and $p_{ij}, g_i \in K$ $(1 \leq i, j \leq n)$ for some differential field K. The independent variable is t, and the unknown functions are $x_1(t), \dots, x_n(t)$. A solution to this system consists of a set $\{x_1(t), \dots, x_n(t)\}$ that satisfies the system for all values of t in some interval $a < t < b$.
An initial value problem is a system as above together with n initial conditions

$$x_1(t_0) = X_1, \quad x_2(t_0) = X_2, \dots , x_n(t_0) = X_n \tag{7}$$

for some t_0 in (a, b).

We abbreviate (6) and (7) to

$$x'(t) = F(t, x)$$

and

$$x(t_0) = X,$$

where

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad F = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}, \quad \text{and} \quad X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}.$$

It is no restriction to consider only first-order differential equations since ordinary differential equations of arbitrary order can be expressed as systems of first-order differential equations. We demonstrate this technique with an example:

Example 9
Consider the second-order equation

$$y''(t) + f_1(t)y'(t) + f_2(t)y(t) + g(t) = 0.$$

By writing $x_1(t) = y(t)$ and $x_2(t) = y'(t)$, we get

$$\begin{cases} \frac{\partial x_1}{\partial t} = x_2(t) \\ \frac{\partial x_2}{\partial t} = -f_1(t)x_2(t) - f_2(t)x_1(t) - g(t) \end{cases}$$

Using the above notation the problem of upper triangular linear systems of differential equations can be rewritten as

$$x' = P(t)x + g$$

where

$$P(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) & \dots & p_{1n}(t) \\ & p_{22}(t) & \dots & p_{2n}(t) \\ & & \ddots & \vdots \\ & & & p_{nn}(t) \end{pmatrix}$$

is an upper triangular matrix with coefficients $p_{ij}(t) \in K$ (K a differential field).

We use the method of back substitution to solve an upper triangular system. First note that the equation for the function $x_n(t)$ does not involve the other functions and can be solved directly as a first-order linear equation by the following method described in [Gra97]:

We want to solve

$$x_n' = p_{nn}(t)x_n + g_n(t).$$

We solve for $g_n(t)$, multiply both sides by the integrating factor[1]

$$\exp\left(-\int p_{nn}(t)dt\right)$$

and get

$$\exp\left(-\int p_{nn}(t)dt\right) g_n(t) = \exp\left(-\int p_{nn}(t)dt\right)(x_n' - p_{nn}(t)x_n).$$

The integrating factor was chosen to give

$$\frac{d}{dt}\left(\exp\left(-\int p_{nn}(t)dt\right)x_n\right) = \exp\left(-\int p_{nn}(t)dt\right) g_n(t).$$

Integrating both sides and dividing by the integrating factor, we get:

$$x_n(t) = \frac{1}{\exp\left(-\int p_{nn}(t)dt\right)}\left(\int \exp\left(-\int p_{nn}(t)dt\right) g_n(t)dt + C\right)$$

which solves the differential equation. Since the p_{ij} are elements of a differential field K we can apply the algorithm of Rothstein/Trager to find an extension field of K that contains a solution if such a solution exists.

Once we have obtained the formula for $x_n(t)$, we can substitute it into the equation for $x_{n-1}(t)$, which reads

$$x_{n-1}'(t) = p_{n-1n-1}(t)x_{n-1}(t) + p_{n-1n}(t)x_n(t) + g_{n-1}(t).$$

The last two terms on the right-hand side are known, so this equation can be solved using the integrating factor

$$\exp\left(-\int p_{n-1n-1}(t)dt\right).$$

By continuing this procedure we obtain the remaining functions $x_{n-1}(t), \dots, x_1(t)$.

Thus any upper triangular system of linear differential equations can be solved by successively solving each of a series of first-order linear equations.

[1] This factor is chosen such that the left-hand side is the derivative of a product. See [Gra97] p. 39 for more details.

4.2 Further Applications

The field of differential algebra has been very active during the past decades. Many important results have been proved and work is still progressing. In this paper we showed some central techniques and their application to a narrow class of differential equations. To solve general systems of linear differential equations it is necessary to generalize the concept of elementary functions to Liouvillian functions. This comprises a thorough introduction into more advanced concepts of Galois theory and exceeds the bounds of this paper. We refer the reader to the original publications by Singer [Sin91].

Appendix

Definition 4.2 (Resultant)
Let $A(x), B(x) \in R[x]$ be nonzero polynomials with $A(x) = \sum_{i=0}^{m} a_i x^i$ and $B(x) = \sum_{i=0}^{n} b_i x^i$.
The Sylvester matrix of A and B is the matrix $M \in R^{(m+n)\times(m+n)}$

$$M = \begin{pmatrix} a_m & a_{m-1} & \cdots & a_1 & a_0 & & \\ & a_m & a_{m-1} & \cdots & a_1 & a_0 & \\ & & \cdots & \cdots & \cdots & \cdots & \\ & & & a_m & \cdots & \cdots & a_0 \\ b_n & b_{n-1} & \cdots & b_1 & b_0 & & \\ & b_n & b_{n-1} & \cdots & b_1 & b_0 & \\ & & \cdots & \cdots & \cdots & \cdots & \\ & & & b_n & \cdots & \cdots & b_0 \end{pmatrix}$$

where the upper part of the matrix consists of n rows of coefficients of $A(x)$, the lower part consists of m rows of coefficients of $B(x)$, and the entries not shown are zero.

The resultant of $A(x)$ and $B(x) \in R[x]$ (written $\mathrm{res}(A, B)$) is the determinant of the Sylvester matrix of A, B. We also define $\mathrm{res}(0, B) = 0$ for nonzero $B \in R[X]$, and $\mathrm{res}(A, B) = 1$ for nonzero coefficients $A, B \in R$. We write $\mathrm{res}_x(A, b)$ if we wish to include the polynomial variable.

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