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Classification of algebraic surfaces

The algebraic surfaces presented by the algebraic equations with three variables which describe surfaces in the Cartesian system of co-ordinates are investigated. Factors at variables can accept any positive and negative material values. The exponent of variables is the whole positive number.

Primary classification of algebraic surfaces is offered on the maximum sedate indicator (an equation order).

The surface of 1st order is described by the equation

$a_3x + a_2 y + a_1 z + a_0 = 0.$

The surface of 2nd order is described by the equation

$a_9x^2 + a_8xy + a_7xz + a_6y^2 + a_5yz + a_4z^2 + a_3x + a_2y + a_1z + a_0 = 0.$

The surface of 3rd order is described by the equation

$a_{19}x^3 + a_{18}x^2y + a_{17}x^2z + a_{16}xy^2 + a_{15}xyz + a_{14}xz^2 + a_{13}y^3 + a_{12}y^2z + a_{11}yz^2 + a_{10}z^3 + a_9x^2 + a_8xy + a_7xz + a_6y^2 + a_5yz + a_6z^2 + a_3x + a_2y + a_1z + a_0 = 0.$

The surface of 4th order is described by the equation

$a_{34}x^4 + a_{33}x^3y + a_{32}x^3z + a_{31}x^2y^2 + a_{30}x^2yz + a_{29}x^2z^2 + a_{28}xy^3 + a_{27}xy^2z + a_{26}xyz^2 + a_{25}xz^3 + a_{24}y^4 + a_{23}y^3z + a_{22}y^2z^2 + a_{21}yz^3 + a_{20}z^4 + a_{19}x^3 + a_{18}x^2y + a_{17}x^2z + a_{16}xy^2 + a_{15}xyz + a_{14}xz^2 + a_{13}y^3 + a_{12}y^2z + a_{11}yz^2 + a_{10}z^3 + a_9x^2 + a_8xy + a_7xz + a_6y^2 + a_5yz + a_6z + a_3x + a_2y + a_1z + a_0 = 0.$

The surface of n th order is described by the equation

$a_mx^n + a_{m-1}y^n + a_{m-2}z^n + \dots a_0 = 0,$

Where m - number of factor of the algebraic equation, important, received depending on an order of the equation [1]. The variety of values of factors of the algebraic equation defines the geometrical form of a surface, its scaling, orientation and placing in spatial system of co-ordinates.

In process of increase in an order of the equation the number of factors of the equation and every possible forms of surfaces (tab. 1) grows.

Table 1

Degree of the algebraic equation	Number of factors
1	4
2	10
3	20
4	35
5	56
6	84
7	120
8	165
9	220
10	286

It is necessary to notice, that at sign material representation of values of factors of the algebraic equation any change of separate factors of the equation creates the new form of a surface. As theoretically material numbers can have values from a minus to infinity plus also the number of forms of surfaces of any order can be infinite. Only in special cases change of one factor or grooms of factors will lead to turn, moving in space or to a stretching or compression on one or several axes. In the general case any change even one factor lead to formation of the new form.

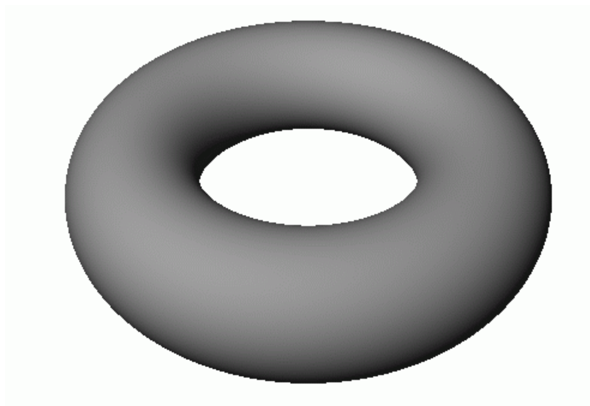
If equation factors to change slightly it is possible to carry out smooth transition from one form of a surface to another. It is caused by properties of the algebraic equations: smoothness and a continuity.

Further it is possible to classify surfaces, dividing them on types, subtypes and kinds.

Types of surfaces

The surface type is certain by conditions at which numerical values of factors of the equation define the characteristic geometrical form of a surface. As an example we will result torahs which is surface that is formed in movement. It is received by circle moving to space on a trajectory at which the center of this circle itself passes on a circle which lays in a plane, a perpendicular plane containing an initial circle.

The equation тора: $x^4 + y^4 + z^4 + 2x^2y^2 + 2x^2z^2 + 2y^2z^2 - 10x^2 - 10y^2 + 3z^2 + 9 = 0$



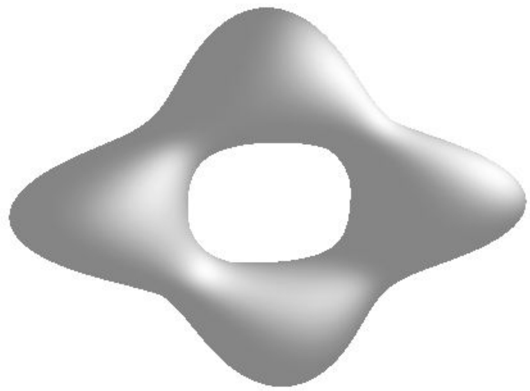
Torahs

Subtypes of surfaces

Surface subtypes - the changed surface keeps the basic properties (similarity, similarity) surface type.

For example, in drawing 25 it is represented torahs with factor (a_{31}) at x^2y^2 equal 6. We name such subtype of a surface «modified torahs 1»

The equation modified torahs 1: $x^4 + y^4 + z^4 + 6x^2y^2 + 2x^2z^2 + 2y^2z^2 - 10x^2 - 10y^2 + 3z^2 + 9 = 0$



Torahs

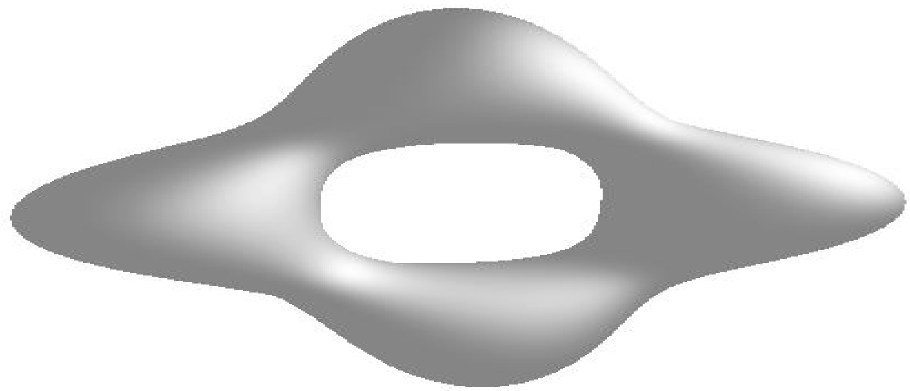
As we see, there is a similarity which consists that this surface has an aperture in the middle, however if at classical тора at its section a plane containing the central axis always will be two circle of identical radius in the given figure in section there will be two closed not crossed flat curves.

Kinds of surfaces

Surface kind - type or a subtype of the surface proportionally changed in sizes concerning three axes of system of co-ordinates (scaling), turned or moved to space.

For example: the Equation of a kind of a surface «modified torahs 1, stretched in one and a half time along axis X»:

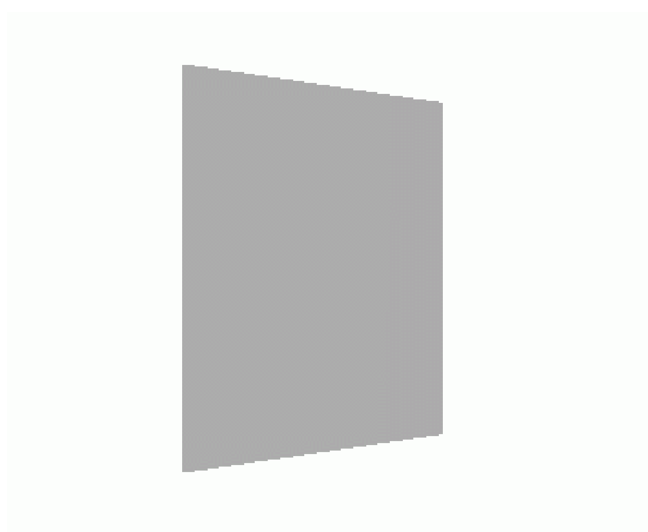
$$\frac{1}{1.5^4} x^4 + y^4 + z^4 + \frac{6}{1.5^2} x^2y^2 + \frac{2}{1.5^2} x^2z^2 + 2y^2z^2 - \frac{10}{1.5^2} x^2 - 10y^2 + 3z^2 + 9 = 0$$



Modified torahs 1, extended on axis X in 1.5 times

Classification of types and subtypes of 1st and 2nd usages

The analysis of the algebraic equations of 1st order has shown, that we have only one type of a surface - a plane. Any changes of factors of the equation of the first degree lead to turn or plane moving, define a kind of a plane and do not create new types and subtypes of a surface of 1st order. Always there is an equation decision.

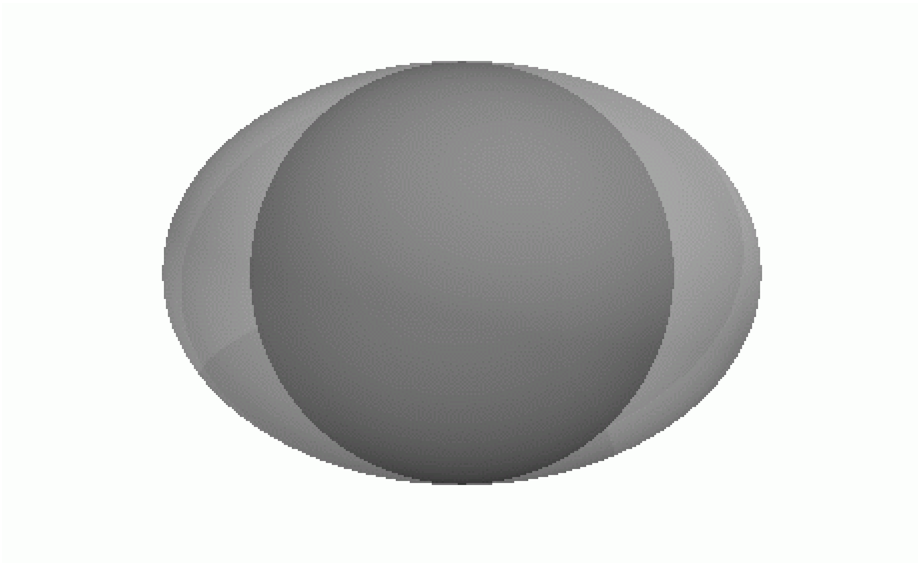


Surface of the first order

The analysis of the algebraic equations of 2nd order shows: there were various types of surfaces and so-called imaginary surfaces when change of one factor leads to absence of the decision of the algebraic equation. An example of an imaginary surface: $x^2 + y^2 + z^2 = -1$

Classification by types of surfaces of 2nd order starts with invariance of factors of the equations of 2nd order: two planes; spherical, cylindrical, conic, hyperbolic, parabolic surfaces.

Classification by subtypes of surfaces of 2nd order it is carried out, forming the geometrical form of a surface by change of the sizes of an initial surface on one or two axes of system of co-ordinates. For example, stretching sphere on one of axes of system of co-ordinates, we receive two-axes ellipsoid .



The stretched sphere

In the equation it leads to change of value of factor at a corresponding variable. However here it is necessary to notice, that change of one factor in the equation in one case to equivalently joint change of several factors at turn of axes of co-ordinates. Under “equivalently” identity of resultants of surfaces here is understood.

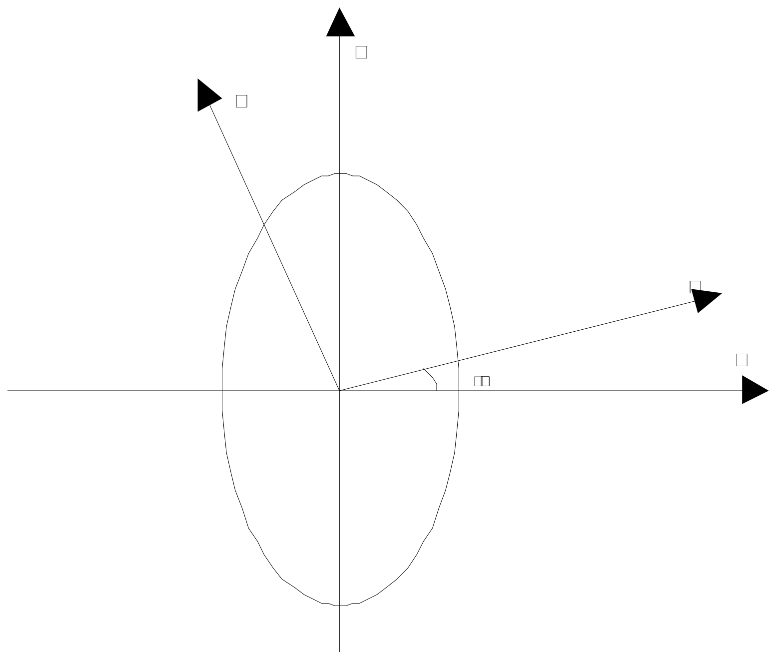
Let's consider a simple example:

Let is ellipsoid :

$4x^2 + y^2 + z^2 = 1$

Let's write down this equation simply as a set of factors: a_9=4 a_6=1 a_3=0 a_0=-1

Let's consider other axes of co-ordinates which are received by turn initial round axis Z on 30 degrees



Ellipse in two axes of co-ordinates

In new axes same ellipsoid registers so:

$$3\frac{1}{4}x^2 + \frac{\sqrt{3}}{2}xy + y^2 + z^2 = 1$$

Let's write down this equation as a set of factors:

$a_9 = 3\frac{1}{4}$ $a_8 = \frac{\sqrt{3}}{2}$ $a_6 = 1$ $a_3 = 0$ $a_0 = -1$

Now we will stretch this ellipsoid to sphere. In the first equation the factor a_9, and in the second - factors a_9 and a_8 will change only.

Surfaces of 2nd order are subdivided into following subtypes.

Two planes: merging, parallel, crossed, imaginary.

Sphere: two-axes ellipsoid ; three-axes ellipsoid , imaginary.

The cylindrical: circular, elliptic, hyperbolic, parabolic, imaginary.

The conic: the circular; elliptic, imaginary.

The hyperbolic: circular with one strip, elliptic with one strip,
circular with two strips , elliptic with two strips.
The parabolic: circular, elliptic, hyperbolic.
For surfaces of 2nd order 6 types and 22 subtypes are defined.

Reception of kinds of surfaces

For reception of kinds of surfaces of any usages it is necessary to use transformation of factors of their equations under formulas:

Scaling on three axes

$x_m = x * K_m$;; $y_m = y * K_m$; $z_m = z * K_m$,

Where K_m - scaling number.

Carrying over to space

$x_d = x + dx$; $y_d = y + dy$; $z_d = z + dz$,

Where dx, dy, dz - sizes of carrying over of a surface on corresponding axes.

Turn in space

Round an axis x

$x_r = x$; $y_r = y \cos r - z \sin r$; $z_r = y \sin r + z \cos r$;

Round an axis y

$y_r = y$; $y_r = x \cos r - z \sin r$; $z_r = x \sin r + z \cos r$;

Round an axis z

$z_r = z$; $x_r = x \cos r - y \sin r$; $y_r = x \sin r + y \cos r$.

Using the given formulas it is possible to receive any quantity of kinds of surfaces from types and subtypes.

Method of search of values of factors of the equation (-1,0,1)

The analysis of surfaces of 2nd order shows, that the equations of the surfaces which axes of symmetry are located or pass through the beginning of system of co-ordinates, have the values of factors equal on occasion or-1 or 0 or 1. It is accepted to name such equations resulted. There is a basis to assume, that we can receive types of surfaces of the higher order systems of co-ordinates symmetric rather the beginning at substitution in the general algebraic equation of the fourth order of values of factors equal or-1 or 0 or 1. As the general search assumes every possible combinations from-1, 0, 1, number of such searches on 3 for each of 35 factors of the equation of 4th order to equally very big number 50 031 545 098 999 707. However it does not mean, that there is such number of every possible types of surfaces of the fourth order.

Let's consider ways of reduction of number of searches.

If in the equation of the fourth order factors with a_{34} on a_{20} we will receive the equation of the third order and if with a_{34} on a_{10} - the equation of 2nd order and if with a_{34} on a_4 - the equation of 1st order are zero. It means, that for surfaces of the fourth order all factors with a_{34} on a_{20} should not be equal to zero simultaneously.

If we move a surface in space form-building factors will not change the value, therefore it is possible to be limited only to change of these factors. Lowering calculations of definition of these factors, we name these factors: with a_{34} on a_{20}. At these factors total degree of variables is equal to a surface order. In our case it is equal 4. At surface turn in space round three axes on 90^0 we receive 9 positions of a surface in space with change only signs on form-building factors. These variants too need to be excluded.

The factor a_0 is the equalizing factor and gives the chance to define imaginary surfaces (when the equation has no decisions).

These conclusions are fair for surfaces of any order (except the first).

It is possible to assume, that changes in factors at variables where total degree of variables below a surface order, will lead to creation only subtypes or kinds of surfaces.

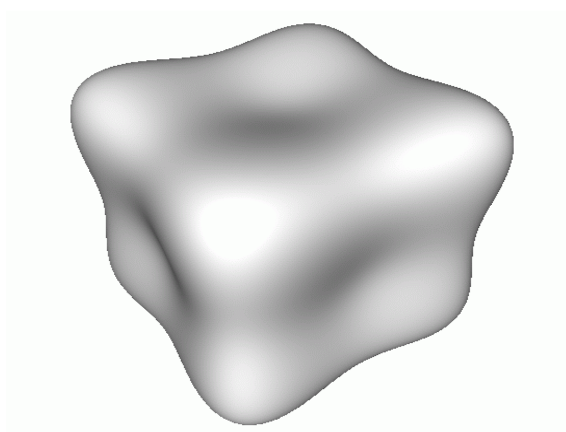
Thus, we have reduced to search of 11 factors for a surface of 3rd order (with a_{19} on a_{10}, a_0) and 16 factors for a surface of 4th order (with a_{34} on a_{20}, a_0).

The number of possible types of surfaces of 3rd order, including imaginary surfaces and excepting zero variants, can be 59 047, and for surfaces of 4th order 4 782 967.

Simple enough kinds of the equations can give interesting forms of surfaces. For example, the equation

$$x^4 + y^4 + z^4 - x^2 - y^2 - z^2 = 0$$

Gives a surface very similar to a human tooth (fig. 10).



Surface "Tooth"

3. Conclusions

The received results show, that difficult geometrical forms can be written down compactly in the form of the numbers which are in factors of the algebraic equation of a surface.

Creation of libraries of difficult surfaces opens ample opportunities of use of descriptions of objects of the live and lifeless nature for their transfer on communication channels.

The described methods of construction of new surfaces allow to fill up and expand constantly library of surfaces as the variety of forms of surfaces of the fourth order is very great. During too time even small on number of types of surfaces the library can actively be used in programs of the automated modeling and designing.