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Robust Constrained Model Predictive Control by Linear Matrix Inequalities

Final Report by

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Chapter 1

Introduction

Model predictive control (MPC) is an online control scheme which iteratively computes locally optimal control inputs based on the model of the plant. The main disadvantage of the MPC is that it cannot be able of explicitly dealing with plant model uncertainties. For confronting such problems, several Robust Model Predictive Control (RMPC) techniques has been developed in recent decades. This report introduces a Robust Model Predictive Control technique by using Linear Matrix Inequalities (LMIs), which is mainly presented in [KBM96].

The introduction of MPC and the problem statement are described in the following of this chapter. Chapter 2 firstly illustrates the min-max approach for handling uncertainties, then shows the LMIs methods without and with constraints, respectively. Finally, the summary is given.

1.1 Model predictive control (MPC)

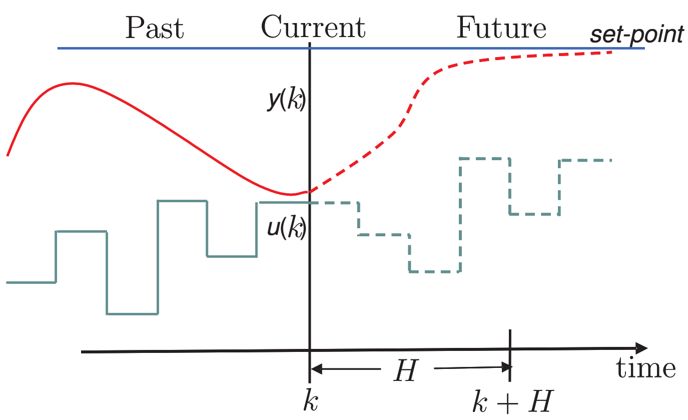

Figure 1.1: Scheme of Model Predictive Control (MPC).

Model predictive control is a well-established online control strategy which iteratively computes locally optimal control inputs by solving an optimization problem over a moving time horizon, (more details in [CB03]). Fig. 1.1 depicts the MPC scheme, where k is the current time, H is the time horizon, y and u stand for the input and output variables, repestively. At current time k, based on the model of the plant, a sequence of the optimal control inputs $u(k), u(k+1), ..., u(k+H)$ is computed by solving an optimization problem over the prediction horizon H meanwhile satisfying input and output constraints. But only the current one is applied to the plant. The optimization is repeated in each time point, i.e. $k+1, k+2, ...$ until the output $y(k)$ reaches the setpoint.

Linear discretized model : the model of the plant can be nonlinear, but here a simple model, namely linear discretized model is introduced here.

$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k)$$

where, x, u, and y are the state, input, and output variables, respectively, A, B, and C are the system matrices with no uncertainties, and k is the current time.

Cost function : cost function can be formulated in different forms. In the report, the quadratic cost function is considered.

$$J(k) = \sum_{i=0}^{H} (x(k+i)^T Q_1 x(k+i) + u(k+i)^T R u(k+i))$$

subject to constraints on the control inputs $u(k+i)$, states $x(k+i)$, and outputs $y(k+i)$, where i is the time index, H is the time horizon, $Q_1 > 0$ and $R > 0$ are the symmetric weighting matrices,
In addition, MPC presents many advantages over other methods, which are:

- it has the capable of dealing with constraints,
- it can easily deal with multivariable case,
- it is an easy-to implement control law,
- it can compensate small disturbances and small model inaccuracies.

1.2 Problem statement

MPC has its powerful abilities and has been applied to many real industrial plant, especially for chemical process. However, MPC has the inability of explicitly handling model plant uncertainties. Hence, several Robust Model Predictive Control

(RMPC) techniques have been developed, i.e. analysis of robustness properties of MPC [GMM82, GMM85a, GMM85b], particle filters [BBOW07], and MPC with explicit uncertainty description [ZMM93].

The report introduces only the MPC with explicit uncertainty description, which modifies the on-line constrained minimization problem to a min-max problem, namely minimizing the worst-case value of the objective function, where the worst case is taken over the set of uncertain models. The detail information will be introduced in the next Chapter.

Chapter 2

Robust Model Predictive Control (RMPC)

As described in the last Chapter, there are several methods of dealing with uncertainties on the parameters. This report only describes the min-max approach which finds an upper bound (worst-case value) of the cost function by maximizing it over the uncertainties (bounded), then minimizes the upper bound to generate a sequence of optimal control inputs.

2.1 Min-max approach for RMPC

The min-max approach, also minimax, searches an upper bound by maximizing the cost function under consideration of the bounded uncertainties, then computes the optimal solution by minimizing the upper bound. It transfers the uncertainties from the cost function to the constraints in order to simplify he online computational complexity. Following sections illustrate the problem formulations, i.e. model for uncertain systems, modification of the cost function by deriving an upper bound, and the motivation of the Linear Matrix Inequalities (LMIs) approach.

2.1.1 Models for uncertain systems

Model for the uncertain systems considered in this report is a linear time-varying (LTV) system:

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)u(k) \\ y(k) &= C(k)x(k) \\ [A(k)\; B(k)] &\in \Omega \end{aligned}$$

where, $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^{n_u}$, and $y \in \mathbb{R}^{n_y}$ are the state, input, and output variables, respectively, and k is the current time.

The uncertainties of the system are defined on system matrics $A(k)$ and $B(k)$ with a prespecified set (a polytope) Ω.

$$\Omega = Co\{[A_1\, B_1], [A_2\, B_2], ..., [A_L\, B_L]\}$$

where Co devotes to the convex hull and L is the number of the vertices.

The polytopic system model can be developed as below. Sets of input/output data are given at different operating points or at different times. A number of linear models are developed from each data set. Alternatively, a nonlinear discrete time-varying system $x(k+1) = f(x(k), u(k), k)$ is linearized in the polytope Ω by using Jacobian matrix $[\partial f/\partial x\ \partial f/\partial u]$.

2.1.2 Derivation of the upper bound

The quatratic cost function is defined as below.

$$J(k) = \sum_{i=0}^{H} (x(k+i)^T Q_1 x(k+i) + u(k+i)^T R u(k+i))$$

By using min-max approach the optimization problem is formulated as follow.

$$\min_{u(k+i), i=0,1,...,H} \Big(\max_{[A(k+i)\, B(k+i)] \in \Omega, i \geq 0} J(k) \Big)$$

Instead of using $\max_{[A(k+i)\, B(k+i)] \in \Omega, i \geq 0} J(k)$, an upper bound $V(x)$ can be generated according to the following procedure.

1. Given a quadratic function $V(x) = x^T P x, P > 0$ with $V(0) = 0$,
2. Suppose V satisfies the following inequality:

   $V(x(k+i+1)) - V(x(k+i)) \leq -[x(k+i)^T Q_1 x(k+i) + u(k+i)^T R u(k+i)]$

   for $x(\infty) = 0$, $V(x(\infty)) = 0$,
3. Sum the equation from $i = 0$ to $i = \infty$, then $-V(x(k)) \leq -J(k)$ can be computed, which indicates:

$$\max_{[A(k+i)\, B(k+i)] \in \Omega, i \geq 0} J(k) \leq V(x(k))$$

Therefore, an upper bound is derived and the original optimization problem can be reformulated as:

$$\min_{u(k+i), i=0,1,...,H} V(x(k))$$

which still implicitly depends on the uncertainties. This problem leads to the optimization involving Linear Matrix Inequalities (LMIs), by which a constant upper bound is derived. Then this upper bound is minimized by a constant state-feedback control law $u(k+i) = Fx(k+i), i \geq 0$.

In addition, there are two main reasons why LMI optimization is relevant to MPC.

- The optimization problems can be solved in polynomial time by using LMI.
- Robust control problems can be recasted in to LMI formulations.

The introduction of LMIs and LMI-based optimization will be shown in the following.

2.1.3 Linear Matrix Inequalities (LMIs)

A linear matrix inequality or LMI is a matrix inequality of the form:

$$M(x) = M_0 + \sum_{r=1}^{m} s_r M_r > 0$$

where $s \in \mathbb{R}^m$ is the variable, and $M_r = M_r^T \in \mathbb{R}^{n \times n}$. The multiple LMIs $M_1(x) > 0, ..., M_n(x) > 0$ can be expressed as a single LMI: $diag(M_1(x), ..., M_n(x)) > 0$. For more details, [BGFB94] is referred.

Convex quadratic inequalities are converted to LMIs form using Schur complements. Given $Q(x) = Q(x)^T$, $R(x) = R(x)^T$ and $S(x)$ depend affinely on x, then the LMI:

$$\begin{aligned} \begin{bmatrix} Q(x) & S(x) \\ S(x)^T & R(x) \end{bmatrix} > 0 &\Leftrightarrow R(x) > 0, Q(x) - S(x)R(x)^{-1}S(x)^T > 0 \\ &\Leftrightarrow Q(x) > 0, R(x) - S(x)^T Q(x)^{-1} S(x) > 0 \end{aligned}$$

From Schur complement, LMI-based optimization can be formulated as:

$$\begin{aligned} &\min c^T x \\ &s.t.\ M(x) > 0 \end{aligned}$$

where M is a symmetric matrix that depends affinely on the optimization variable x, and c is a real vector of appropriate size.

2.2 Linear Matrix Inequalities (LMI) Approach for RMPC

After introducing the LMI-based optimization and the problem formulation of the uncertainty system, linear matrix inequalities (LMIs) approach for robust model predictive control (RMPC) is described in this Chapter. The main concept of the LMI approach is that at each time instant, an LMI optimization problem (as opposed to conventional linear or quadratic programs) is solved that incorportates input and output constraints and a description of the plant uncertainty, and guarantees certain robustness properties [KBM96].

2.2.1 Robust unconstrained MPC

After substituting by the upper bound, the original optimization problem can be formulated as:

$$\min_{u(k+i),i=0,1,...,H} V(x(k))$$

Suppose that the uncertainty set Ω. Then given the state feedback matrix F in the control law $u(k+i) = Fx(k+i),\ i \geq 0,\ F = YQ^{-1}$, where $Q > 0$ and Y is obtained from the solution of the following linear minimization problem:

$$\min_{\gamma,Q,Y} \gamma$$

$$s.\ t.\ \begin{bmatrix} 1 & x(k)^T \\ x(k) & Q \end{bmatrix} \geq 0$$

$$\begin{bmatrix} Q & QA_j^T + Y^T B_j^T & QQ_1^{1/2} & Y^T R^{1/2} \\ A_j Q + B_j Y & Q & 0 & 0 \\ Q_1^{1/2} Q & 0 & \gamma I & 0 \\ R^{1/2} Y & 0 & 0 & \gamma I \end{bmatrix} \geq 0$$

where $j = 1, 2, ..., L$, L is the vertices number of the convex hull Ω. The proof can be checked from Appendix A in [KBM96].

The feed back matrix F: $u(k+i) = Fx(k+i)$ is constant. But in the presence of uncertainty, F shows a strong dependence on the state of the system. In order to avoid the problem, recomputing $F(k+i)$ at each sampling time is used. (The significant improvement in performance as opposed to using a static state feedback control law can be found in [KBM96].)

2.2.2 Robust constrained MPC

Using *Lemma 1* (Invariant ellipsoid), the input and output constraints can be formulated as below. The proof can be seen in Appendix B in [KBM96].

Input constraints Given $||u(k+i)||_2 \leq u_{max},\ i \leq 0$, from [BGFB94] and using Schur complement, the LMI:

$$\begin{bmatrix} u_{max}^2 I & Y \\ Y^T & Q \end{bmatrix} \geq 0$$

holds at all times $i \leq 0$.

Output constraints At sampling time k, consider

$$\max_{[A(k+j)\ B(k+j)]\in\Omega, j\geq 0} ||y(k+i)||_2 \leq y_{max},\ i \geq 0$$

the following LMI:

$$\begin{bmatrix} Q & (A_jQ+B_jY)^TC^T \\ C(A_jQ+B_jY) & y_{max}^2 I \end{bmatrix} \geq 0$$

holds for $j = 1, 2, ..., L$.

Finally, the original optimization problem can be formulated as a LMI-based optimization problem. The cost function is transferred as below.

$$\min_{u(k+i), i=0,1,...,H} (\max_{[A(k+i)\ B(k+i)]\in\Omega, i\geq 0} J(k))$$

$$\Rightarrow \min_{u(k+i), i=0,1,...,H} V(x(k))$$

$$\Rightarrow \min_{\gamma, Q, Y} \gamma$$

$$s.\ t. \begin{bmatrix} 1 & x(k)^T \\ x(k) & Q \end{bmatrix} \geq 0$$

$$\begin{bmatrix} Q & QA_j^T + Y^TB_j^T & QQ_1^{1/2} & Y^TR^{1/2} \\ A_jQ+B_jY & Q & 0 & 0 \\ Q_1^{1/2}Q & 0 & \gamma I & 0 \\ R^{1/2}Y & 0 & 0 & \gamma I \end{bmatrix} \geq 0$$

$$\begin{bmatrix} u_{max}^2 I & Y \\ Y^T & Q \end{bmatrix} \geq 0$$

$$\begin{bmatrix} Q & (A_jQ+B_jY)^TC^T \\ C(A_jQ+B_jY) & y_{max}^2 I \end{bmatrix} \geq 0$$

Chapter 3

Numerical Example

An angular positioning system [KS72] shown in Fig 3.1 is described in this Chapter, which illustrates the developed algorithm. The control aim is to drive the motor pointing always to the target object.

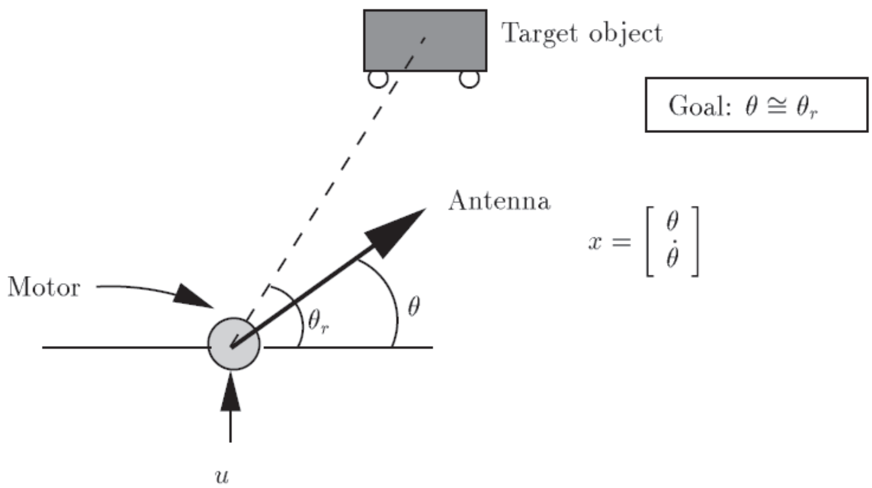

Figure 3.1: Angular positioning system.

System dynamics Following is the system dynamics:

$$x(k+1) = \begin{bmatrix} \theta(k+1) \\ \dot{\theta}(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 - 0.1\alpha(k) \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0.1\kappa \end{bmatrix} u(k)$$

$$y(k) = [1 \; 0]x(k)$$

with $\kappa = 0.787$, $0.1 \leq \alpha(k) \leq 10$, and $x(0) = [0.05, 0]^T$, where $\alpha(k)$ is proportional to the coefficient of viscous friction.

Since $0.1 \leq \alpha(k) \leq 10$, the uncertainty set Ω is defined as $A(k) \in \Omega = Co\{A_1,\ A_2\}$, where

$$A_1 = \begin{bmatrix} 1 & 0.1 \\ 0 & 0.99 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 0.1 \\ 0 & 0 \end{bmatrix}$$

Cost function : the cost function is defined as:

$$\min_{u(k+i)=Fx(x+i),i\geq 0} \left(\max_{A(k+i)\in\Omega,i\geq 0} J(k) = \sum_{i=0}^{H} (y(k+i)^2 + 0.00002u(k+i)^2) \right)$$
$$s.t. ||u(k+i)||_2 \leq 2, i \geq 0$$

The software LMI control toolbox [GNLC95] was used to compute the solution of the linear objective minimization problem. From Fig. 3.2, by using robust LMI-based MPC, the system becomes stable. In addition, Fig. 3.3 shows that the performance of using varying state feedback matrix $F(k)$ is 4 times faster than the static state feedback.

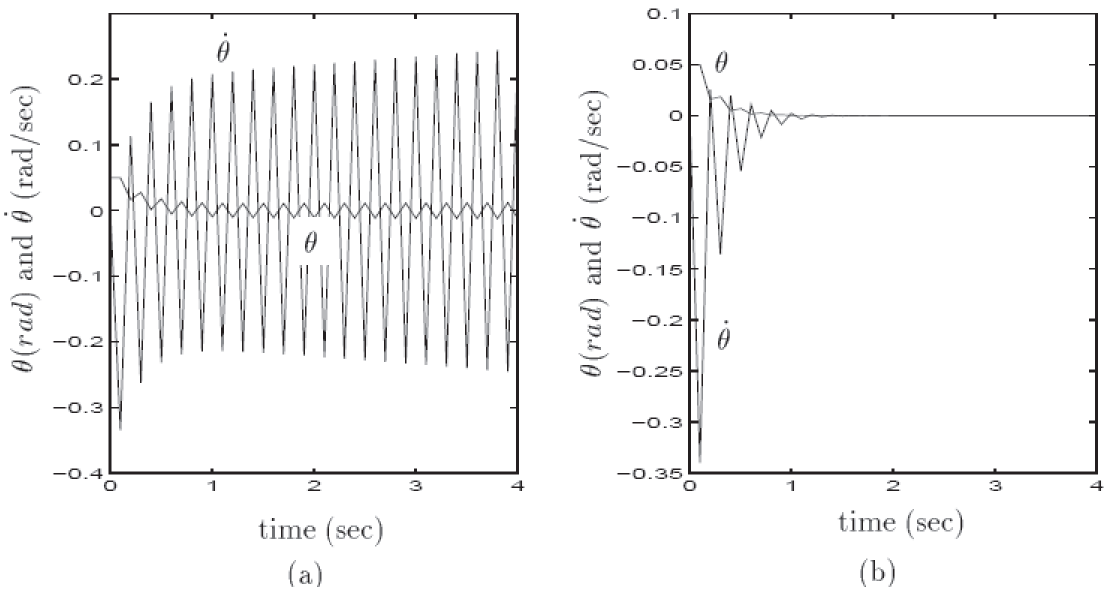

Figure 3.2: Unconstrained closed-loop responses for the plant: (a) using standard MPC $\alpha(k) = 1$; (b) using robust LMI-based MPC.

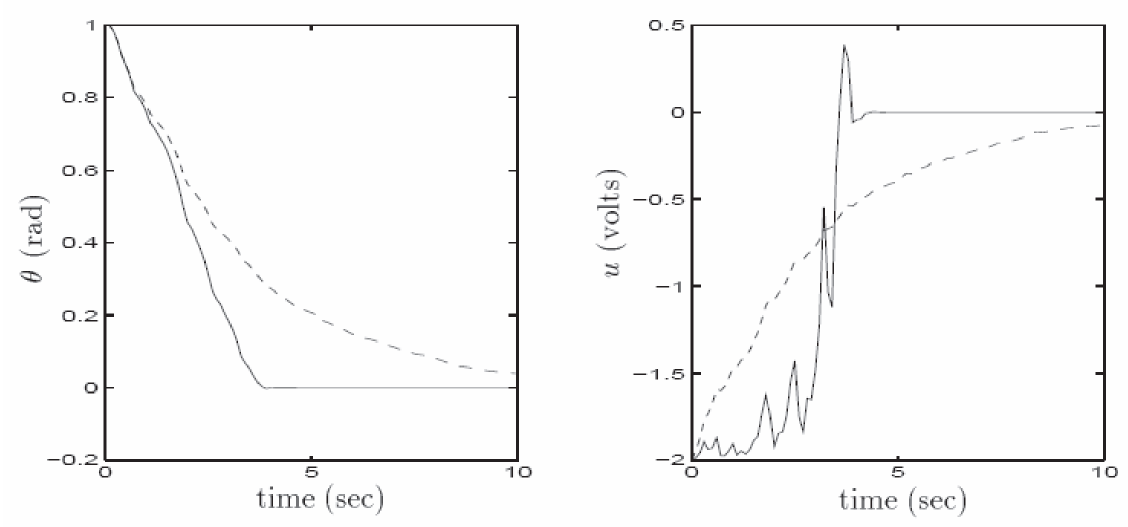

Figure 3.3: Closed-loop responses for the time-varying system with input constraint: solid lines, using robust receding horizon state feedback; dashed lines, using robust static state feedback.

Chapter 4

Conclusions

A new theory for robust constrained model predictive control synthesis, based on the assumption of full state feedback, was introduced by using linear matrix inequalities. The numerical example shows the ability of the developed algorithm for explicitly dealing with model plant uncertainties. The new approach can handle models with additive uncertainties, reference trajectory tracking, delay problems. The robust model predictive control for hybrid systems becomes the one of the main research direction currently. The detail information of the whole approach can be found in [KBM96].

List of Figures

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