Introduction to the Theory of Complexity Classes and Logic

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Part I

Complexity Theory

Contents

Languages and Turing-machines
- Turing-machines
- Nondeterminism

Complexity Classes
- *P* and *NP*
- Space Complexity
- Relation between various Complexity Classes
- Completeness

Outline

Languages and Turing-machines
Turing-machines
Nondeterminism

Complexity Classes
P and *NP*
Space Complexity
Relation between various Complexity Classes
Completeness

Language

Definition (Language)

- Σ denotes a finite, non-empty set called alphabet; its elements are called symbol
- For symbols $l_1, l_2, \ldots, l_m \in \Sigma$ we write $l_1 l_2 \cdots l_m$ for its concatenation; concatenated symbols are called words
- The set of all words constructed by concatenating symbols in $\Gamma \subseteq \Sigma$ is denoted by Γ^*
- A subset $L \subset \Sigma^*$ is called language

Definition (Language)

- For a word w we write $|w|$ for its length.
- Shorthand notation:
  - The concatenation of words $w_1 = l_1 \cdots l_m$, $w_2 = k_1 \cdots k_n$ is defined as $w_1 w_2 := l_1 \cdots l_m k_1 \cdots k_n$.
  - Let Γ be a set of words. Then Γ^* denotes the set of all words constructed by concatenating the words in Γ.
  - For a letter a we write a^* instead of $\{a\}^*$; similarly we write w^* instead of $\{w\}^*$ for a word w.

Turing-machines

Definition (Turing-machine)

A (deterministic) Turing-machine M is a quadruple $M = (Q, \Sigma, \delta, s)$, where Q is a finite set of *states* and Σ is an alphabet containing the special symbols $\sqcup$, the *blank symbol*, and $\triangleright$, the *first symbol*; $s \in Q$ is the *initial state*. Q and Σ are disjoint.

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A Turing-machine can be seen as a cursor on a string of symbols and having a definite internal state. Every time the cursor moves over the end of the string a new $\sqcup$ is inserted on the string.
The function δ is the "program" of the machine. It specifies, for each combination of the current state $q \in Q$ and current symbol $a \in \Sigma$, a triple $\delta(q, l) = (p, b, D)$. p is the next state, b is the symbol, which is written over a, and $D \in \{\rightarrow, \leftarrow, -\}$ is the direction in which the cursor will move.

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By definition a Turing-machine always starts on the symbol $\triangleright$. It cannot overwrite $\triangleright$ and will always upon reading $\triangleright$ move to the right.

Input of a Turing-machine

A Turing-machine M may be initialized with a word x written on its string. This word x is called *input* of M and for the resulting Turing-machine we write $M(x)$.

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If M does not halt on input x, we write $M(x) = \nearrow$.

Example

$M = (Q, \Sigma, \delta, s)$, where $Q = \{s\}$, $\Sigma = \{0, 1, \sqcup, \triangleright\}$ and δ as in the table below. We examine the behavior of M with the input 101:

$q \in Q$	$l \in \Sigma$	$\delta(q, l)$
s	$\triangleright$	$(s, \triangleright, \rightarrow)$
s	0	$(s, 1, \rightarrow)$
s	1	$(s, 0, \rightarrow)$
s	$\sqcup$	$(h, \sqcup, —)$

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4. s, $\triangleright010\underline{\sqcup}$
5. h, $\triangleright010\underline{\sqcup}$

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1. s, $\triangleright\underline{1}01$
2. s, $\triangleright 0\underline{0}1$
3. s, $\triangleright 01\underline{1}$
4. s, $\triangleright 010\underline{\sqcup}$
5. h, $\triangleright 010\underline{\sqcup}$

So the output is $M(101) = 010$.

Definition (Configuration)

A configuration of a Turing-machine $M = (Q, \Sigma, \delta, s)$ contains all information of the current state of M. Formally it is a triple (q, w, u), where $q \in Q$ is the current state of M, w is the word on the left of the cursor including the symbol scanned by the cursor and u is the word on the right.

In the example M is at step 3. in the configuration $(s, 011, \epsilon)$.

Definition (Yields in one step)

Let M be a Turing-machine. We say that a configuration (q, w, u) yields the configuration (q', w', u') in one step, denoted by $(q, w, u) \xrightarrow{M} (q', w', u')$, if a step of the machine from configuration (q, w, u) results in the configuration (q', w', u').

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Formally, it means that the following holds: Let l be the last symbol of w and $\delta(q, l) = (q', l', D)$. If $D = \rightarrow$, then w' is w with l replaced by l' and appended by the first symbol of u; u' is u with its first symbol removed. If $D = \leftarrow$, then w' is w with l' omitted from its end and u' is $l'u$. Finally if $D = —$, then w' is w with l replaced by l and $u' = u$.

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In the example $(s, \triangleright, 101)$ yields $(h, \triangleright 010, \sqcup)$.

Definition (Deciding and accepting languages)

Let $L \subset (\Sigma \setminus \{\sqcup\})^*$ be a language and M be a Turing-machine. If the output of M for any word x is either “yes”, when $x \in L$, or “no”, when $x \notin L$ then we say, that M decides L and L is called a *recursive language*.

If $M(x) =$“yes”, when $x \in L$, and $M(x) = \nearrow$, when $x \notin L$, then we say M simply accepts L and L is called *recursively enumerable*.

Definition (Computation of functions)

Let $f : (\Sigma \setminus \{\sqcup\})^* \to \Sigma^*$ be function and M be a Turing-machine with alphabet Σ. We say that M computes f if, for any word $x \in (\Sigma \setminus \{\sqcup\})^*$, $f(x) = M(x)$. Then f is called a *recursive function*.

For example the function from $\{0, 1\}^*$ to $\{0, 1, \sqcup\}$, that replaces 0 with 1 and vice versa until it encounters a $\sqcup$, is recursive by the example.

Definition (k-string Turing-machines)

A k-string Turing-machine is a quadruple $M = (Q, \Sigma, \delta, s)$, where Q, Σ and s are exactly as in ordinary Turing-machines. But here δ is a function from $Q \times \Sigma^*$ to $(Q \cup \{h, \text{“yes”}, \text{“no”}\}) \times (\Sigma \times \{\leftarrow, \rightarrow, —\})^k$. We still assume that $\triangleright$ is at the start of each string, cannot be overwritten and upon reading it the cursor must move to the right.
The input of M is written on the first string and the output on the last string.

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The input of M is written on the first string and the output on the last string.

Intuitively, δ decides the next state of the machine by looking at its current state and the symbol at the cursor of each band. It then overwrites the current symbol on each band by another and moves the cursors on each band either left or right or not at all.

Analogous to 1-string Turing-machines we define *configuration* and *yields* for k-string Turing-machines:
A *configuration* is a $k+2$ tuple, with the first coordinate being the current state, the even ones being the word on the left of the cursor, the odd ones the word on the right of the cursor.
Yields is defined exactly as in 1-string machines after taking the greater number of strings into account.

Example (Palindromes)

We now construct a 2-string Turing-machine, that decides the language of palindromes on $\Sigma = \{0, 1\}$. It first copies its input on the second string, then it moves the first cursor on the first symbol of the input and the second on the last symbol. Then it moves the two cursors in opposite direction comparing the two symbols at the cursors.

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There is no Turing-machine significantly faster at deciding palindromes than this one.

Example (Palindromes)

$q \in Q$	$l_1 \in \Sigma$	$l_2 \in \Sigma$	$\delta(q, l_1, l_2)$
s	$\triangleright$	$\triangleright$	$(s, \triangleright, \rightarrow, \triangleright, \rightarrow)$
s	0	$\sqcup$	$(s, 0, \rightarrow, 0, \rightarrow)$
s	1	$\sqcup$	$(s, 1, \rightarrow, 1, \rightarrow)$
s	$\sqcup$	$\sqcup$	$(q, \sqcup, \leftarrow, \sqcup, —)$

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s	$\sqcup$	$\sqcup$	$(q, \sqcup, \leftarrow, \sqcup, -)$
q	0	$\sqcup$	$(q, 0, \leftarrow, \sqcup, -)$
q	1	$\sqcup$	$(q, 1, \leftarrow, \sqcup, -)$
q	$\triangleright$	$\sqcup$	$(p, \triangleright, \rightarrow, \sqcup, \leftarrow)$

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s	$\sqcup$	$\sqcup$	$(q, \sqcup, \leftarrow, \sqcup, —)$
q	0	$\sqcup$	$(q, 0, \leftarrow, \sqcup, —)$
q	1	$\sqcup$	$(q, 1, \leftarrow, \sqcup, —)$
q	$\triangleright$	$\sqcup$	$(p, \triangleright, \rightarrow, \sqcup, \leftarrow)$
p	0	0	$(p, 0, \rightarrow, 0, \leftarrow)$
p	1	1	$(p, 1, \rightarrow, 1, \leftarrow)$
p	0	1	$(\text{“no”}, 0, \rightarrow, 1, \leftarrow)$
p	1	0	$(\text{“no”}, 1, \rightarrow, 0, \leftarrow)$
p	$\sqcup$	$\triangleright$	$(\text{“yes”}, \sqcup, \rightarrow, \triangleright, \leftarrow)$

Nondeterminism

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A nondeterministic Turing-machine is a quadruple $N = (Q, \Sigma, \delta, s)$, where Q, Σ and s are as in deterministic Turing-machines. But now δ is a function from $Q \times \Sigma$ to $\mathcal{P}((Q \cup \{h, \text{“yes”}, \text{“no”}\}) \times \Sigma \times \{\leftarrow, \rightarrow, —\})$. That is, for a nondeterministic machine there may be more than one appropriate next step or none at all. It can also be in multiple configurations at once.

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A computation of a nondeterministic machine can be imagined as a computation tree where each path can be computed by a deterministic machine.

Definition (Yields for nondeterministic Turing-machines)

Similarily we define (q, w, u) yields (q', w', u') in one step as with deterministic machines except that only one $(q', l', D) \in \delta(q, l)$ must match the definition of yields in one step for deterministic machines.

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Yields in k steps and *yields* are defined exactly as with deterministic machines.

Definition (Deciding languages for nondeterministic Turing-machines)

We say N decides a language L, if for any $x \in \Sigma^*$ the following is true: $x \in L$ if and only if $(s, \triangleright, x)$ yields $(\text{“yes”}, w, u)$ for some words w and u.

This definition of decision sets the nondeterministic machine apart from the deterministic one: An input is accepted if there is *one* computation path that results in “yes”.

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P and *NP*
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P and *NP*

We now introduce now the notion of the time needed to compute an output.

Definition (TIME($f(n)$))

For an input x of a k-string Turing-machine M the time required by M on input x is simply the number of steps to halting. If M does not halt on x the time required is ∞.

P and NP

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Now let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a function. We say the M operates within time $f(n)$ if, for any input x, the time required by M on x is at most $f(|x|)$. f is then called a time bound for M.

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Finally let $L \subset (\Sigma \setminus \{\sqcup\})^*$ be language decided by M operating in time $f(n)$. Then we say that $L \in \mathrm{TIME}(f(n))$.

That is, TIME($f(n)$) contains exactly those languages that can be decided by a Turing-machine within time bound $f(n)$.

Example (Palindromes)

In our example we constructed a Turing-machine M, which decides the language of palindromes L on $\{0, 1\}$. Now let us count how many steps M needs in the *worst case* to accept a word x with length n.

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At first the machine copies x to the second band and moves the first cursor to the first symbol needing $2n + 3$ steps. Then is compares the symbols of the first band with the symbols on the second band needing n steps.
Altogether M needs at most $3n + 3$ steps and as such $L \in \mathrm{TIME}(3n + 3)$.

Definition (P)

P is the set of all languages decidable by a Turing-machine in a polynomial time bound. That is:

$$P = \bigcup_{k=0}^{\infty} \mathrm{TIME}(n^k)$$

The number of strings does not significantly increase the speed of Turing-machines.

Theorem (Simulating a k-string Turing machine with 1 string)

Given any k-string Turing machine operating within time bound $f(n)$, there is a 1-string Turing-machine M' operating within time $\mathcal{O}(f(n)^2)$, such that $M(x) = M'(x)$ for any input x.

This theorem also holds for nondeterministic Turing-machines.

Now analogous to the definition of TIME and P:

Definition (NTIME($f(n)$))

We say that a nondeterministic Turing-machine N *decides* a language L in time $f(n)$, where $f : \mathbb{N} \rightarrow \mathbb{N}$, if N decides L and for all words x with length n no computation path may be longer than $f(n)$.

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Then NTIME($f(n)$) is the set of languages decidable by a nondeterministic Turing-machine within time bound $f(n)$.

Now analogous to the definition of TIME and P:

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Then NTIME($f(n)$) is the set of languages decidable by a nondeterministic Turing-machine within time bound $f(n)$.

Note that we do not tax non-deterministic machines for the whole amount of computation going on; just for the longest path.

Definition (*NP*)

NP is the set of all languages decidable by a nondeterministic Turing-machine within a polynomial time bound. That is:

$$NP = \bigcup_{k=0}^{\infty} \mathrm{NTIME}(n^k)$$

NP does not depend on the number of strings of the Turing-machines.

Obviously it holds that $P \subset NP$: The deterministic Turing-machines are exactly those nondeterministic machines where $|\delta(q, l)| = 1$ for all q and l.

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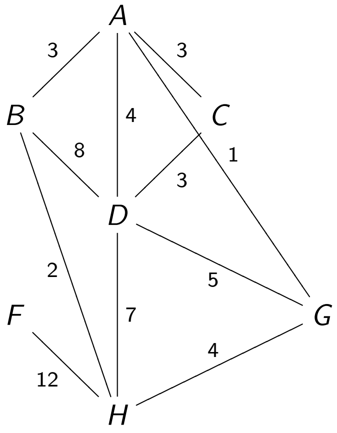
It is strongly conjected to be not the case.

Example (Traveling Salesman Problem)



Given a weighted, possibly directed graph, what is the shortest path such that each vertice is visited at least once. It can be transformed in a decision problem: Given an integer B is there a path with length at most B such that each vertice is visited at least once. We call this problem TSP(D).

Example (Traveling Salesman Problem)

By encoding the problem TSP(D) in an alphabet, we can use Turing-machines to solve it: the most obvious algorithm is calculating all routes and checking if the shortest one is shorter than B or not. If n is the number of vertices, then this takes about $n!$ steps; the best known algorithm is not much better than that.

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But TSP(D) is obviously in NP: a nondeterministic Turing-machine can calculate all routes at once and check if it is shorter than B or not, a task which takes roughly n^2 steps.

We now give a more intuitive characterization of NP.

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Let $R \subset \Sigma^* \times \Sigma^*$ be a relation. R is said to be *polynomially decidable* if there is a deterministic Turing-machine which decides the language $\{x\#y,\ (x,y) \in R\}$ in polynomial time. R is called *polynomially balanced* if $(x,y) \in R$ implies $|y| \leq |x|^k$ for some $k \geq 1$.

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Theorem

Let L be a language. Then $L \in NP$ if and only if there is a polynomially decidable and polynomially balanced relation R, such that $L = \{x,\ (x, y) \in R \text{ for some } y\}$.

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Such a y is called *certificate* for x.
An example for a certificate would be the vertices of a path shorter than B for TSP(D).

Now we can describe P and NP informally:

P contains those languages L, for which $x \in L$ can be decided quickly

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- NP contains those languages L, for which a certificate for $x \in L$ can quickly be verified.

This is another reason why it is believed that $P = NP$ does not hold: It is usually much easier to verify a solution than finding one.

Theorem (Simulating a nondeterministic Turing-machine by a deterministic one)

Let L be language decided by a nondeterministic Turing-machine N in time $f(n)$. Then there is deterministic Turing-machine M deciding L in time $\mathcal{O}(c^{f(n)})$, where $c > 1$ depends on N alone. Or put differently:

$$\mathsf{NTIME}(f(n)) \subset \bigcup_{c>1} \mathsf{TIME}(c^{f(n)})$$

Space Complexity

We now introduce another way of quantifying “complexity”: by the sum of the maximum lengths of the strings.

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We now introduce another way of quantifying “complexity”: by the sum of the maximum lengths of the strings.
To do this properly we have to introduce a special version of Turing-machines: Ones with special input and output bands.
Otherwise we could not study machines which need asymptotically less or equal space for computation than the length of the input respectively output.

Definition (k-string Turing-machine with input and output)

A (non)deterministic k-string Turing-machine, $k \geq 3$, with input and output is a Turing-machine that scans over its input only once, does not overwrite it and stops at the end of the input string. Furthermore it never moves the last cursor to the left.

Definition (Space required)

Let M be a k-string Turing-machine with input x such that the starting configuration $(s, \triangleright, x, \triangleright, \epsilon, \ldots)$ yields $(H, w_1, u_1, \ldots, w_k, u_k)$, where H is halting state. Then the *space required* by M on input x is $\sum_{i=1}^{k} |w_i u_i|$.

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If however M is a Turing-machine with input and output then the first and last band are not counted: the *space required* by M on input x is then $\sum_{i=2}^{k-1} |w_i u_i|$.

Now we can define the deterministic space complexity class just like the time complexity one.

Definition (SPACE($f(n)$))

Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a function and M be a Turing-machine. We say M operates within space bound $f(n)$, if for any input x the space required by M is at most than $f(|x|)$.
We say a language L is in $(f(n))$, if there is a Turing-machine with input and output that decides L in space bound $f(n)$.

Now we define space complexity for nondeterministic machines.

Definition (Deciding in space $f(n)$)

Let $f : \mathbb{N} \to \mathbb{N}$ be a function and N be a k-string nondeterministic Turing-machine with input and output. We say *N decides L in space $f(n)$*, if N decides L and if for any input x this implication holds:

$$(s, \triangleright, x, \triangleright, \epsilon, \ldots) \xrightarrow{N}{}^{*} (q, w_1, u_1, \ldots, w_k, u_k)$$

$$\Rightarrow$$

$$\sum_{i=2}^{k-1} |w_i u_i| \leq f(|x|)$$

So N may not use in any computation path more space than $f(|x|)$. Note that we do not require for N to halt on all computations.

Definition (NSPACE($f(n)$))

Let $f : \mathbb{N} \to \mathbb{N}$ be a function. We say that a language L is in NSPACE($f(n)$), if there is a k-string nondeterministic Turing-machine with input and output that decides L in space $f(n)$.

Definition (Some space complexity classes)

$$PSPACE := \bigcup_{k=1}^{\infty} \mathrm{SPACE}(n^k)$$
$$NPSPACE := \bigcup_{k=1}^{\infty} \mathrm{NSPACE}(n^k)$$

Relation between various Complexity Classes

Theorem (Relation between deterministic and nondeterministic classes)

1. $\mathrm{SPACE}(f(n)) \subset \mathrm{NSPACE}(f(n))$, $\mathrm{TIME}(f(n)) \subset \mathrm{NTIME}(f(n))$

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3. $\mathsf{NSPACE}(f(n)) \subset \mathsf{TIME}(k^{\log(n)+f(n)})$

For any complexity class S we can define $coS = \{L,\ L^{\complement} \in S\}$ as the class of the complements of languages in S.

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Any deterministic complexity class D is closed under complement, i.e. for $L \in C$ its complement $L^{\complement}$ is also in C.

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Any deterministic complexity class D is closed under complement, i.e. for $L \in C$ its complement $L^{\complement}$ is also in C.

Proof.

Let L be a language in C and M be a Turing-machine that decides L within the bound required by the class D. Now let M' be the Turing-machine whose output is “no” whenever the output of M is “yes” and vice versa. Then M decides $L^{\complement}$ in the same bound as M and therefore $L^{\complement} \in D$. □

Theorem (Relation between space complexity classes)

$$PSPACE = NPSPACE$$

$$NPSPACE = coNPSPACE$$

Open questions

It is unknown whether:

- $P = NP$?
- $NP = coNP$?

Both answer are suspected to be “no”, but so far a proof supporting either way remains elusive since the beginning of the theory of complexity classes.

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Various deterministic and nondeterministic complexity classes can be defined analogous to P and NP using other functions than polynomials, e.g. logarithms or exponentionals.
For all of these it is also unknown whether they are equal or not.

Completeness

Definition (Reduction)

Let Σ and Γ be alphabets and $L_1 \subset \Sigma^*$ and $L_2 \subset \Gamma^*$ be languages. We say that L_1 is *reducible* to L_2, if there is a function $f : \Sigma^* \rightarrow \Gamma^*$ computable by a deterministic Turing-machine in logarithmic space such that the following holds for all inputs x: $x \in L_1$ if and only if $f(x) \in L_2$. f is then called *reduction* from L_1 to L_2.

Note that the composition of reductions is also a reduction; it follows that *reducible* is a transitive relation.

Theorem

P, NP, coNP, PSPACE, NPSPACE are closed under reductions, i.e. if $L_2 \in P/NP/coNP/PSPACE/NPSPACE$ and L_1 can be reduced to L_2 then $L_1 \in P/NP/coNP/PSPACE/NPSPACE$.

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Let C be a complexity class and $D \subset C$ be another complexity class closed under reductions. If $L \in C$ is a C-complete language, then $L \in D$ if and only if $C = D$.

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Proof.

"$\Leftarrow$": trivial. "$\Rightarrow$": Let L' a language in C; since $L \in D$ is complete, L' can be reduced to L. The reduction is also in D since D is closed under reductions. Therefore $C = D$. $\square$

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So one way to solve $P \overset{?}{=} NP$ positively, is to search for a *NP*-complete problem and find a polynomial time deterministic algorithm to solve it.

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- PARTITION
- ...

Part II

Propositional Logic

Definition (Syntax of propositional logic)

Let $V = \{x_1, x_2, \ldots\}$ be a countable set of *variables* and C be a finite set of *logical connectives* with arbitrary, finite arity. Then all of the following are *propositional formulas*:

- The signs $\bot$ and $\top$
- Variables
- $\oplus(p_1, \ldots, p_k)$, where $\oplus \in C$ is a logical connective (with arity k) and $p_1, \ldots, p_k$ are propositional formulas.

Parentheses may be added to clarify the precedence of logical connectives.

Definition (Valuation)

A *valuation* $v : V \to \{\top, \bot\}$ is a function that gives variables the values “true” or “false”

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Definition (Propositional language)

Given a set of variable V, a finite set of connectives C and a valuation v a *propositional language* defines the *value* of a propositional formula inductively:

- $\top$ means “true”, $\bot$ means “false”
- Each $x \in V$ means $v(x)$
- For each $k \in \mathbb{N}_0$, each connective $\oplus$ in C and each k-tuple of formulas $(t_1, \dots, t_k)$ each t_i meaning either “true” or “false” it defines whether $\oplus(t_1, \dots, t_k)$ is “true” or “false”

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After fixing a language we can give any formula ϕ a meaning of either “true” or “false” using a valuation v. For this we write $\phi(v)$.

Example (A propositional logic)

Here is a well known propositional language:

$$V = \{x_1, \ldots\},\ C = \{\vee, \wedge, \neg\}$$

Examples of such propositional formulas

$$x_1 \wedge x_2,\ (\neg(x_3 \vee x_1)) \wedge x_1, \ldots$$

With the usual meaning of the connectives this propositional language is called *standard base*. Usually when we talk about propositional formulas, we mean this language.

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With the usual meaning of the connectives this propositional language is called *standard base*. Usually when we talk about propositional formulas, we mean this language.
The other well known symbols $p \Rightarrow q$ and $p \Leftrightarrow q$ can be seen as abbreviations of $\neg p \vee (p \wedge q)$ or $(p \Rightarrow q) \wedge (q \Rightarrow p)$.

The standard way to prove theorems about a propositional formula ϕ is the *induction on the structure of p*. For example:

Theorem

Let ϕ be a propositional formula and v_1 and v_2 be valuations that agree on the set P of variables in ϕ, i.e. $v_1|_P = v_2|_P$. Then $\phi(v_1) = \phi(v_2)$.

Proof.

Proof by induction on the structure of ϕ.

- $\phi = \bot$ or $\phi = \top$:
  $\phi(v) =$"true" respectively "false" regardless of valuation v
- $\phi = x$ with x variable:
  $\phi(v_1) = v_1(x) = v_2(x) = \phi(v_2)$ because v_1 and v_2 agree on $\{x\}$.
- $\phi = \oplus(\tau_1, \dots, \tau_k)$ where τ_i are propositional formulas for which the hypothesis $\tau_i(v_1) = \tau_i(v_2)$ holds:
  $\phi(v_1) = \oplus(\tau_1(v_1), \dots, \tau_k(v_1)) = \oplus(\tau_1(v_2), \dots, \tau_k(v_2)) = \phi(v_2)$

Therefore the hypothesis holds. $\square$

Many proofs about propositional formulas follow the same scheme.

Definition (Satisfiability and Tautology)

A propositional formula ϕ is called

satisfiable , if there is a valuation v such that $\phi(v) =$"true".

The set of all satisfiable formulas is called SAT, the set of all tautology TAUT.

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equivalent	to a formula ψ if $\phi \Leftrightarrow \psi$ is a tautology.

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About SAT and TAUT

SAT is in *NP*: A nondeterministic machine can "guess" all valuations of a formula at once and then check if one is true. This can be done polynomial time.

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It even is *NP*-complete: For any nondeterministic machine N deciding L in polynomial time and any input x, there is a formula F constructible in logarithmic space that is satisfiable if and only if $x \in L$.

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It even is NP-complete: For any nondeterministic machine N deciding L in polynomial time and any input x, there is a formula F constructible in logarithmic space that is satisfiable if and only if $x \in L$.
Instead of asking if $\phi \in$TAUT one can ask if $\neg\phi \in$SAT. In that sense SAT and TAUT are essentially the same.

Outline

Abstract Proof Systems

Example of a Propositional Proof System: Resolution

Proof Systems

Definition ((Abstract) Proof systems)

A *proof system* for a language L is a deterministic Turing-machine M operating in polynomial time that for any word x, there exists a certificate p, such that M accepts $x\#p$ if and only if $x \in L$.

Recall the characterization of NP through certificates: It is clear that the language L is in NP.
In that case, p is called a *proof* for x.
The time bound of the machine M is called *(time) complexity* of the proof.

Definition (Propositional proof system)

A *propositional proof system* is a proof system for the language TAUT of tautologies.
If there exists a proof p in this system for every tautology x, then the system is called *complete*.
If the existence of a proof p for x in this system implies that x is a tautology, then the system is called *sound*.

Since a formula is a tautology if and only if its negation is unsatisfiable, we can give an alternative definition:

Definition (Propositional proof system)

A *propositional proof system* is a proof system for the language UNSAT of unsatisfiable formulas.

Since there is a great number of propositional proof systems, it is useful to have a way of saying that one is as strong as another.

Definition (Simulation)

Let M and M' be two proof systems. We say that M *polynomially simulates* M' if M and M' prove the same language and proofs in M' can be polynomially converted into proofs in M, i.e. there exists a in polynomial time computable function such that p is a proof in M' if and only if $f(p)$ is a proof in M'.

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Definition (Equivalence of proof systems)

Two proof systems M and M' are said to be *equivalent* if M polynomially simulates M' and vice versa.

Outline

Abstract Proof Systems

Example of a Propositional Proof System: Resolution

Examples of Propositional Proof Systems

Now a short introduction to a common proof system: Resolution
The proof systems will be described algorithmically rather than stating the corresponding Turing machine.

Resolution

Definition (Literal, Conjunctive and Disjunctive Normal Form)

A propositional formula ϕ is

a literal if ϕ is a variable or a negated variable.

in conjunctive normal form (CNF) if it is a conjunction of disjunctions of literals.

in disjunctive normal form (DNF) if it is a disjunction of conjunctions of literals.

Any formula has an equivalent formula in CNF and one in DNF.

Definition (Equisatisfiable)

Two formulas ϕ and ϕ' are called equisatisfiable, if ϕ is satisfiable if and only if ϕ' is satisfiable.

Example: $(x \Leftrightarrow p) \wedge (x \vee q)$ and $p \vee q$, with p, q formulas, are equisatisfiable, but not equivalent.

Let ϕ be a formula in CNF represented as a set Φ of sets of literals, e.g. $\phi = (x_1 \vee x_2) \wedge x_3 \wedge \neg x_2$ is represented as $\Phi = \{\{x_1, x_2\}, \{x_3\}, \{x_2\}\}$. Resolution is an algorithm for deciding if $\phi \in$SAT working on that set Φ.

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Example:

$$\{\{a, b\}, \{a, \neg b\}, \{\neg a, b\}, \{\neg a, \neg b\}\}$$
$$\{\{a\}, \{\neg a\}\}$$
$$\emptyset$$

Each step in this example *resolved on* a variable: On the first we resolved on b, on the second on a.
The formula represented by this set of sets is not satisfiable.

Definition (Resolvent)

Let C and D be set of literals representing a disjunction and x be variable. If $x \in C$ and $\neg x \in D$, then the set $(C \setminus \{x\}) \cup (D \setminus \{\neg x\})$ is called the *resolvent* of C and D.

DavisPutnam algorithm

Let S be a set of sets of literals representing a CNF and l be a variable. Define

- $P := \{C \setminus \{l\},\ l \in C \in S\}$

Then the formulas represented by S and S' are equisatisfiable. This is one DavisPutnam step building all possible resolvents.

DavisPutnam algorithm

Let S be a set of sets of literals representing a CNF and l be a variable. Define

- $P := \{C \setminus \{l\},\ l \in C \in S\}$
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- $S' := S_0 \cup \{C \cup D,\ C \in P,\ D \in N\}$

Then the formulas represented by S and S' are equisatisfiable. This is one DavisPutnam step building all possible resolvents.

DavisPutnam algorithm

Let S be a set of sets of literals. Construct S' by one DavisPutnam step from S, S'' by one resolution step from S', $S^{(3)}$ from S'' and so on, each time using a new variable.

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Resolution is therefore a complete and sound proof system for propositional formulas in CNF.