# 1D Quantum Rings

Projektarbeit

von

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# Contents

# 1 Motivation and Introduction

There was a growing theoretical and experimental interest in microscopic objects like quantum rings and quantum dots in the last decades. There are few reasons for this interest, the main of them are:

- The possibility to control the number of particles on the ring and its fluctuations
- The possibility to study artificial atoms and their features since quantum rings and dots show shell structure similar to shell structures of real atoms
- The experimental realization of quantum rings which is possible since the development of semiconductor technology

The goal of this report is to give a simple theoretical introduction to quantum rings, which nevertheless explains some features and to show few experimental results from experiments done by other scientists. The biggest part of the content of this report is cited from other papers, listed in the bibliography.

## 1.1 Quantum rings - introduction

The main physical idea is to create and consider a quasi 1D ring with few to hundreds charge-carriers (electrons) on it. The number of charge-carriers should be well defined and its fluctuations under control of the experimentalist. You can see two pictures of a possible setup in fig. 1 and fig. 2:

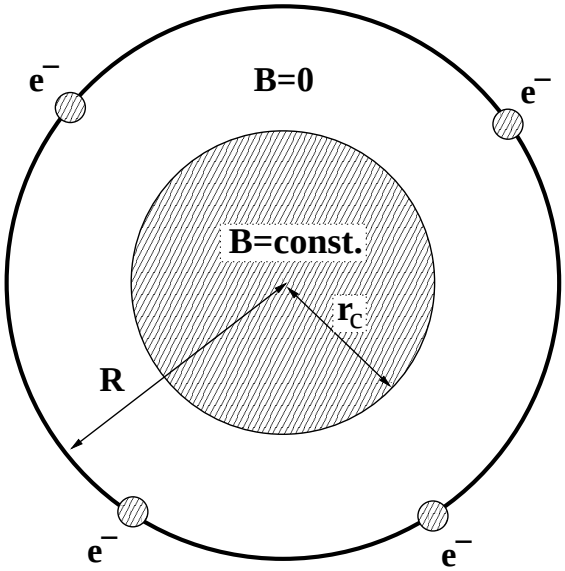



Figure 1: Quantum ring: Schematic setup with an applied magnetic field

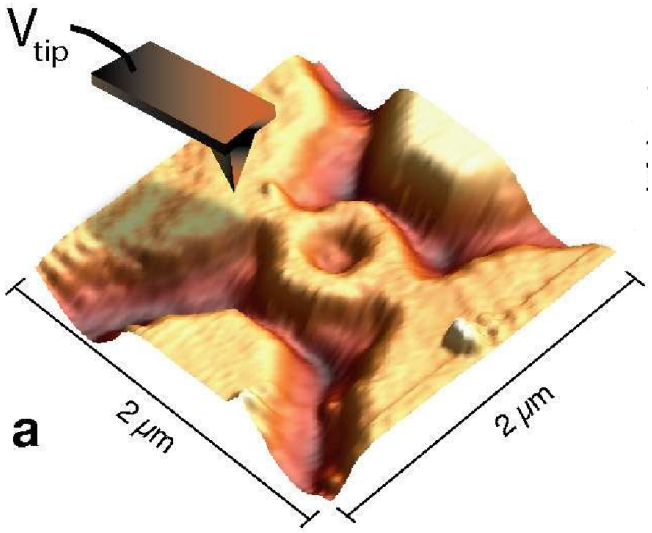



Figure 2: AFM picture of a quantum ring from [1]

Another physical goal was to look at what happens if a magnetic field which penetrates the ring is switched on. If the vector potential is chosen such that the electrons on the ring are field-free but nevertheless there is a flux through the ring some interesting features like the Aharonov-Bohm effect and persistent currents periodic in flux occur. Again, a simple theoretical description and experimental proof of these effects are provided in this report.

## 2 Theoretical approach

In this section a simple theoretical approach to a 1D quantum ring with and without magnetic field is given. Certain assumptions are made:

- Strictly one-dimensional ring filled with electrons
- Non-interacting particles
- Spinless fermions (electrons): only 1 particle per state
- Switching magnetic field on, such that finite flux through the ring, but field-free electrons

Under these assumptions the energy spectrum and the current in the ring are calculated.

## 2.1 Non-interacting particles on a ring

First of all the 1-particle Schrödinger equation has to be solved. The Hamiltonian depends only on the polar angle $\varphi$:

$$H = -\frac{\hbar^2}{2m_e R^2}\frac{\partial^2}{\partial\varphi^2}$$

Which yields the corresponding stationary Schrödinger equation:

$$-\frac{\hbar^2}{2m_e R^2}\frac{\partial^2}{\partial\varphi^2}\Psi_m(\varphi) = E_m\Psi_m(\varphi)$$

and its trivial solution (without normalization):

$$\Psi_m(\varphi) = e^{im\varphi}, E_m = \frac{\hbar^2 m^2}{2m_e R^2}$$

where $R$ is the Radius of the ring, $m_e$ the electron mass and $E_m$ the eigenenergy of the eigenstate $m$. $m \in \mathbb{Z}$ due to single-valueness of the wavefunction $\Psi_m(\varphi + 2\pi) = \Psi_m(\varphi)$ and correnspponds to the angular momentum of the electron.

Since assuming non-interacting particles, the many-body wavefunction of N electrons obeying the Pauli exclusion principle is given by the Slater-determinant:

$$\Psi_{m_1,...,m_N}(\varphi_1,...,\varphi_N) = \begin{vmatrix} \Psi_{m_1}(\varphi_1) & \Psi_{m_2}(\varphi_1) & ... & \Psi_{m_N}(\varphi_1) \\ \Psi_{m_1}(\varphi_2) & \Psi_{m_2}(\varphi_2) & ... & \Psi_{m_N}(\varphi_2) \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \Psi_{m_1}(\varphi_N) & \Psi_{m_2}(\varphi_N) & ... & \Psi_{m_N}(\varphi_N) \end{vmatrix}$$

$$\Psi_{m_k}(\varphi) = \exp(im_k\varphi)$$

where $\Psi_{m_k}(\varphi)$ is given by the single-particle wave-function with angular momentum $m_k$.

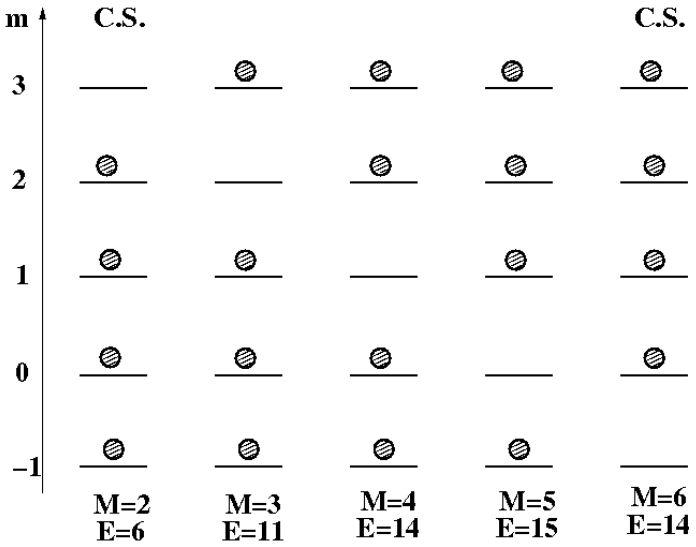
Given eigenfunctions $\Psi_{m_k}(\varphi)$ the total energy can be obtained, it is the sum of the energies of single particles:

$$E = \sum_{i=1}^{N} E_{m_i} = \sum_{i=1}^{N} \frac{\hbar^2 m_i^2}{2m_e R^2}$$

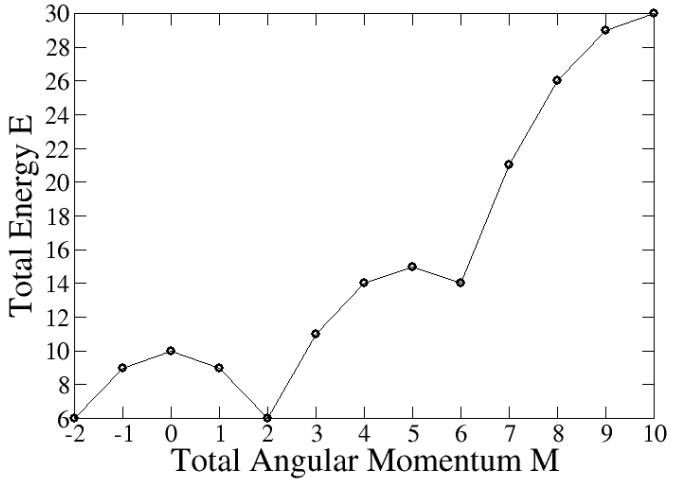
In the same way the total angular momentum $M$ is given by the sum of angular momenta of single particles:

$$M = \sum_{i=1}^{N} m_i$$

According to these formulae one can see, that the lowest possible energy at a given total angular momentum is obtained by occupying the single-particle states (which are given by the angular momentum) as compact as possible, meaning that there should be maximum one empty single-particle state in the many-particle occupation scheme. This is illustrated in fig. 3.




(a) Possible occupation of single particle levels by four electrons. Occupations denoted with C.S. are so called compact states, meaning that the single-particle states are occupied next to each other. For any number of electrons every total angular momentum state can be realized with less than two gaps in the single particle levels.




(b) Lowest possible total energy $E$ of four electrons at a given total angular momentum $M$. The energy is given in relative units and corresponds to $\sum_{i=1}^{N} m_i^2$.

Figure 3: a) One possible occupation scheme of single-particle levels of four electrons and b) Energy spectrum of four non-interacting electrons on a quantum ring

## 2.2 Magnetic field and current: Aharonov-Bohm effect

Now lets have a look at what happens if a magnetic field is switched on. We chose a vector potential such that it is continuos at a critical radius $r_c$ but the magnetic field is constant inside $r_c$ and equal to zero outside:

$$\vec{A} = A_\varphi \cdot \vec{e_\varphi} + A_r \cdot \vec{e_r} + A_z \cdot \vec{e_z}$$

$$A_r = A_z = 0, \qquad A_\varphi = \begin{cases} \frac{B \cdot r}{2}, \text{if } r \leq r_c \\ \frac{B \cdot r_c^2}{2r} = \frac{\Phi}{2\pi r}, \text{if } r > r_c \end{cases}$$

since $\vec{B} = \vec{\nabla} \times \vec{A}$, this results in a magnetic field in z direction (the ring is chosen to be in the x-y-plane)

$$B_z = \begin{cases} B, \text{if } r \leq r_c \\ 0, \text{if } r > r_c \end{cases}$$

while $B_\varphi = B_r = 0$. In fig. 4 you can see a schematic picture of the ring:

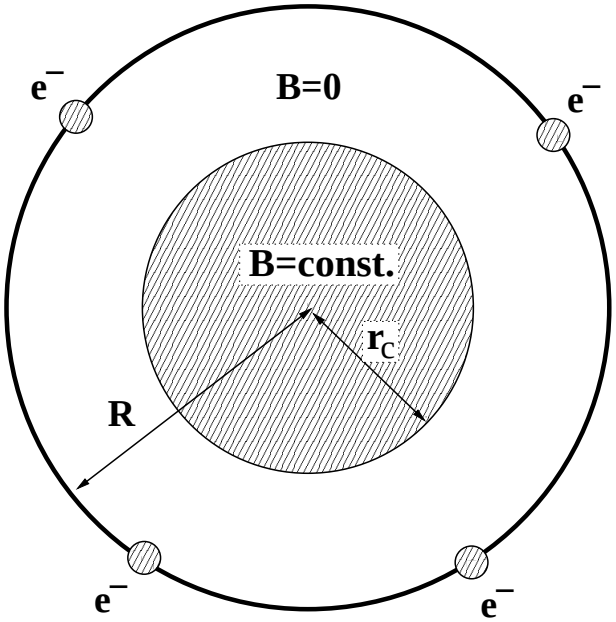



Figure 4: Schematic picture of a quantum ring with magnetic field. The electrons are at $r = R$, meaning they are field-free if $r_c < R$. Nevertheless there is a magnetic flux $\Phi = \pi r_c^2 B$ penetrating the ring.

In the classical picture the electrons are in a field-free area which means that no force acts upon them. Thus, no effect due to changes of the magnetic field are expected classically.

However, quantum mechanically the Schrödinger-equation including the magnetic field has to be solved. The Hamiltonian with magnetic field can be written as

$$\hat{H} = \frac{1}{2m_e}(\hat{p} - \frac{e}{c}\hat{A})^2 = \frac{1}{2m_e}\left(-\frac{i\hbar}{R}\frac{\partial}{\partial\varphi} - \frac{e\Phi}{2c\pi R}\right)^2$$

The eigenfunctions $\Psi_m(\varphi)$ are still the same as for the field-free Hamiltonian, but the eigenenergies changed:

$$\Psi_m(\varphi) = e^{im\varphi}$$

$$E(m,\Phi) = \frac{\hbar^2}{2m_e R^2}\left(m - \frac{\Phi}{\Phi_0}\right)^2 \tag{1}$$

where $\Phi_0 = \frac{hc}{e}$ is the flux quantum.

Obviously the ground-state energy of the spectrum 1 is periodic in flux $\Phi$ with the period of a flux quantum $\Phi_0$: Increasing the flux by one flux quantum forces the electron, in order to stay in the lowest energy state, to change its angular momentum by one. This is illustrated in fig. 5.

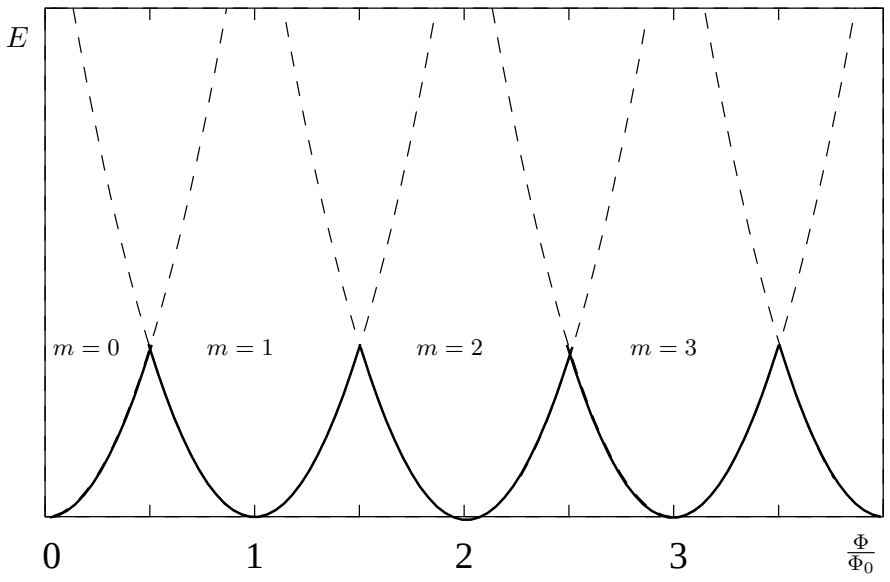



Figure 5: Energy spectrum of an electron on a ring penetrated by a magnetic field. The quadratic functions with different origins represent different angular momenta, according to equation 1. The thick line is the ground state energy.

According to fig. 5 the ground state energy of one electron on a ring penetrated by magnetic field is periodic in flux. In the very same way the ground state energy of several

electrons is also periodic in flux, since each of the electrons stays in the lowest possible not occupied state.

$$E_G\left(\frac{\Phi}{\Phi_0} \pm 1\right) = E_G(\frac{\Phi}{\Phi_0}) \tag{2}$$

Now lets have a look at the current inside the ring. The current operator can be written in the following way:

$$\hat{\vec{j}} = \frac{c}{e}\frac{\partial \hat{H}}{\partial \vec{A}}$$

$$\vec{A} = \frac{\Phi}{2\pi R}\vec{e_\varphi}$$

Thus, the derivative of the energy with respect to flux is given by:

$$\begin{aligned} -c\frac{\partial E}{\partial \Phi} &= -c\left\langle\Psi\right|\frac{\partial \hat{H}}{\partial \Phi}\left|\Psi\right\rangle = -c\left\langle\Psi\right|\frac{\partial \hat{H}}{\partial \vec{A}}\frac{\partial \vec{A}}{\partial \Phi}\left|\Psi\right\rangle = \\ &= \left\langle\Psi\right|\hat{\vec{j}_\varphi}\frac{-e}{2\pi R}\left|\Psi\right\rangle = \frac{-e}{2\pi R}\int_0^{2\pi} d\varphi j(\varphi) = I \end{aligned}$$

The current inside the ring is given by the derivative of energy with respect to flux:

$$I = -c\frac{\partial E}{\partial \Phi} \tag{3}$$

This means that, according to equations 2 and 3, the current produced by electrons in the ground state is also periodic in flux (see fig. 6)!

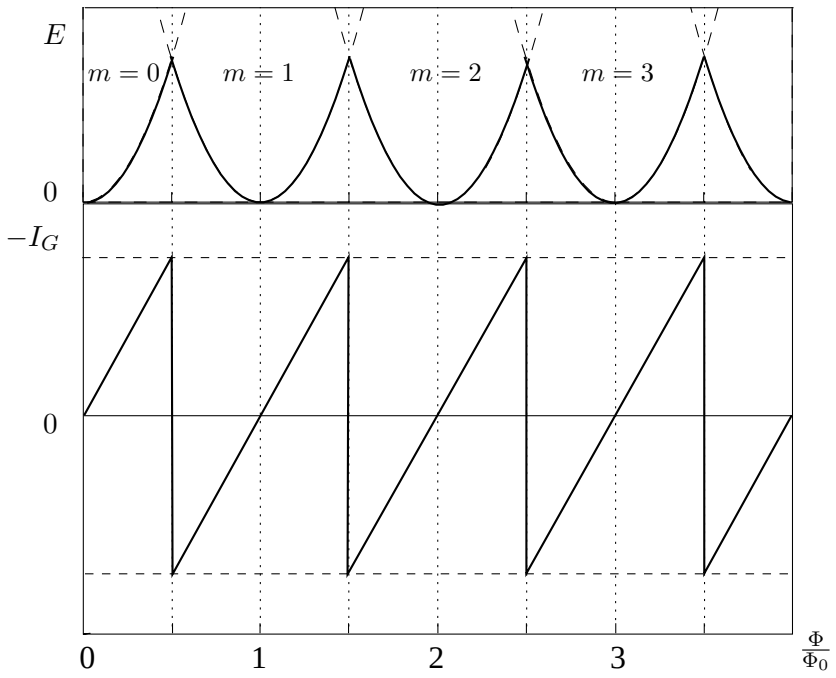



Figure 6: Energy (upper diagram) and current (lower diagram) as functions of flux $\Phi$. Since, according to equation 3, the current is given by the negative derivative of the energy, it is also periodic in flux with the period $\Phi_0$

This is the quantum mechanical current, not the dissipative one. It means that this current goes forever and no heat is produced by it.

Alternatively one may chose a unitary gauge transformation $\hat{U}(\varphi)$ to solve the Schrödinger equation with magnetic field:

$$\hat{U}(\varphi) := \exp\left(\frac{-ie}{c\hbar}\int_0^{\varphi} \vec{A}\cdot d\vec{l}\right) = \exp\left(\frac{-ie}{c\hbar}\frac{\Phi\varphi}{2\pi}\right)$$

which transforms the Hamiltonian to a field-free Hamiltonian in the following way:

$$\hat{H} \to \hat{H}' = \hat{U}\hat{H}\hat{U}^{-1} = -\frac{\hbar^2}{2m_e R^2}\frac{\partial^2}{\partial\varphi^2}$$

Lets apply it on the linear Schrödinger equation:

$$\begin{aligned} \Psi \to \Psi' &= \hat{U}\Psi \\ \hat{H} \to \hat{H}' &= \hat{U}\hat{H}\hat{U}^{-1} \end{aligned}$$

This yields that for a given eigenfunction $\Psi$ with $\hat{H}\Psi = E\Psi$

$$\hat{H}'\Psi' = \hat{U}\hat{H}\hat{U}^{-1}\hat{U}\Psi = E\Psi'$$

the eigenvalues are conserved. Since the transformed Hamiltonian is

$$\hat{H}' = -\frac{\hbar^2}{2m_e R^2}\frac{\partial^2}{\partial\varphi^2}$$

the solutions $\Psi'(\varphi)$ for the corresponding Schrödinger equation are the already known eigenfunctions $\Psi'(\varphi) = \exp(ik\varphi)$. Now $k \in \mathbb{Z}$ isn't necessary since the boundary conditions changed. If periodic boundary conditions were true for the original wavefunction

$$\Psi(\varphi + 2\pi) = \Psi(\varphi)$$

then "twisted boundary conditions" are true for the new wavefunction:

$$\Psi'(\varphi + 2\pi) = \hat{U}(\varphi + 2\pi)\Psi(\varphi + 2\pi) = \Psi'(\varphi) \cdot \exp(\frac{-ie}{c\hbar}\Phi)$$

Thus, if $\Psi'(\varphi) = \exp(ik\varphi)$ ($k \in \mathbb{Z}$ not necessary):

$$\Psi'(\varphi + 2\pi) = \Psi'(\varphi)\exp(ik2\pi) \stackrel{!}{=} \Psi'(\varphi) \cdot \exp(\frac{-ie}{c\hbar}\Phi)$$

$$\Rightarrow 2\pi k = -\frac{e}{c\hbar}\Phi + 2\pi m$$

where $m \in \mathbb{Z}$ now. This condition yields $k = -\frac{e}{2\pi c\hbar}\Phi + m$. Then $\Psi'(\varphi)$ and $\Psi(\varphi)$ are

$$\Psi'(\varphi) = \exp(i\varphi(m - \frac{e\Phi}{c\hbar 2\pi}))$$

$$\Psi(\varphi) = \hat{U}^{-1}\Psi(\varphi) = \exp(i\varphi(m - \frac{e\Phi}{c\hbar 2\pi}) + \varphi\frac{ie\Phi}{c\hbar 2\pi}) = \exp(im\phi)$$

Which is the already known solution resulting in a periodic ground state energy.

Due to the gauge transformation of the wavefunction and the twisted boundary conditions the wavefunction picks up a phase $\Delta$ while moving around the ring from $\varphi_1$ to $\varphi_2$:

$$\Delta = -\frac{ie\Phi}{\hbar 2\pi}(\varphi_2 - \varphi_1) = -i\frac{\Phi}{\Phi_0}(\varphi_2 - \varphi_1) \tag{4}$$

Now imagine a transport current along the circumference of the ring from point A to point B, see fig. 7.

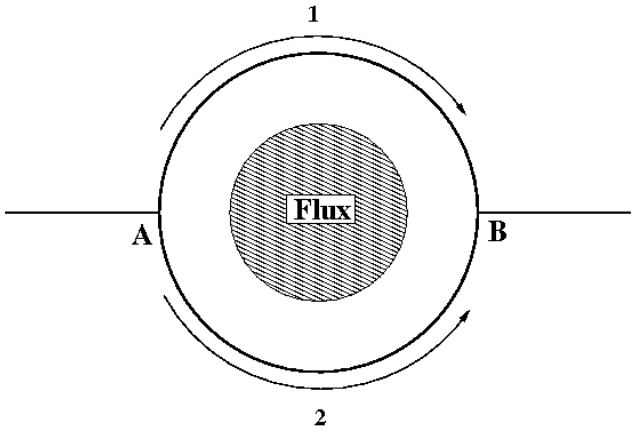



Figure 7: Transport current along the circumference of a quantum ring from A to B

The electrons can either take the upper branch of the ring (path 1 on the picture) or the lower branch (path 2). In the first case the wavefunction picks up the phase $\Delta_1 = -i\frac{\Phi}{\Phi_0}\pi$, in the second case the phase is $\Delta_2 = i\frac{\Phi}{\Phi_0}\pi$. The phase difference of the wavefunctions at point B is $\Delta = 2\pi i\frac{\Phi}{\Phi_0}$, meaning that the phase difference can be tuned via changing the applied magnetic field. Both wavefunctions interfere at point B:

$$\left|\Psi \cdot \exp\left(2\pi i\frac{\Phi}{\Phi_0}\right) + \Psi \cdot \exp\left(-2\pi i\frac{\Phi}{\Phi_0}\right)\right|^2 = 2\left|\Psi\right|\left(1+\cos\left(\frac{\Phi}{\Phi_0}2\pi\right)\right) \tag{5}$$

Thus, the transport current through a quantum ring penetrated by magnetic flux is periodic in flux with the period $\Phi_0$ due to interference of the wavefunctions.

## 2.3 Coulomb blockade

In order to study quantum rings properly one may need to experimentally control several parameters:

- Number of electrons on the ring
- Shape and size of the ring

The mechanisms of controlling these parameters of quantum rings are quite the same as for quantum dots, which are described by Leo Kouwenhoven and Charles Marcus in their paper "Quantum dots", see [2]. The purpose of this section is to explain the mechanism of the so called Coulomb blockade, which is responsible for the stability of the number of electrons on the ring, and the mechanism of controlling the effective shape and size of the ring. These mechanisms are explained on the example of quantum dots and, as already mentioned, are the same for quantum rings. The rest of the section is partially cited from [2].

Here is a schematic picture of a quantum dot, fig. 8.

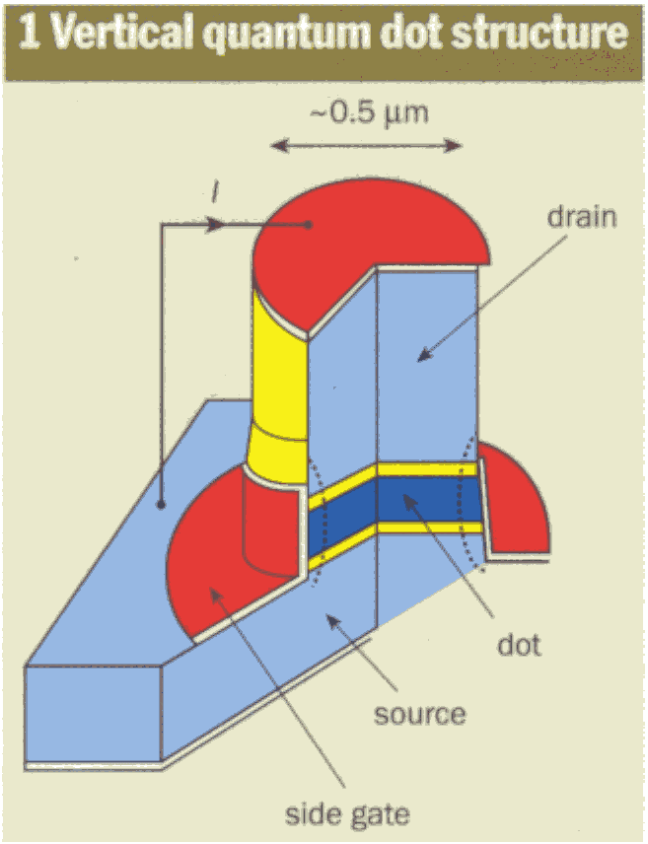



Figure 8: Schematic picture of a quantum dot. Yellow areas (surrounding the dot) are made of insulating material, red areas (gates) are metallic, bright blue areas (source, drain) are semiconducting.

A constant source-drain voltage is applied in order to have a current through the dot. The shape and the size of the dot are tuned via the side gate voltage. Applying a negative side gate voltage has two effects:

- The whole energy of the electrons inside the dot changes
- If we approximate the potential inside the dot as a harmonic potential, the curvature of the harmonic potential is changed via the side gate voltage. Thus, the level spacing of the harmonic oscillator also changes

Due to these two effects electrons can be squeezed out of the dot or sucked inside the dot. In this way, the number of electrons in the dot can be controlled.

This mechanism of controlling the number of electrons is the also called Coulomb blockade: Given a certain number $N$ of electrons on the dot, it takes some energy to overcome the Coulomb repulsion in order to bring one more electron on the dot. This amount of additional energy $E_{add}$ can be described by a simple model, the constant-interaction model:

- The Coulomb interaction between the electrons is independent of $N$ and given by the capacitance $C$ of the dot
- Additional energy $\Delta E$ is needed, which is the difference of the quantum levels of the dot
- Therefore $E_{add} = \frac{e^2}{C} + \Delta E$ is the energy amount needed to place an electron on the dot

Though the assumption that the Coulomb interaction term is $\frac{e^2}{C}$ constant and independent of number of electrons is a rough one this simple model describes quite accurately the diagram for the current inside the dot shown in fig. 9.

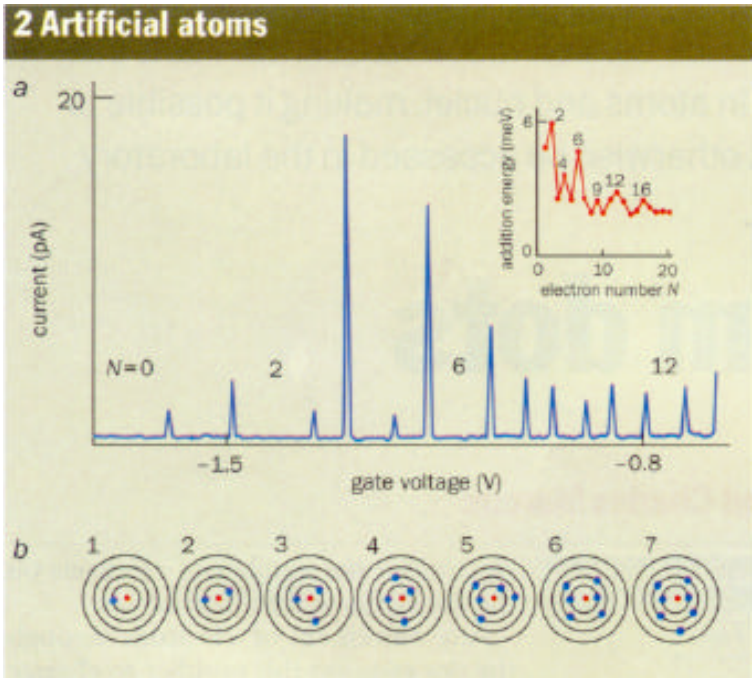



Figure 9: a) Current inside the dot versus the side gate voltage: each peak corresponds to an electron entering the dot b) Shell structure of the dot which explains the higher additional energy for the 3rd, the 7th the 13th electrons

It is of interest, that the amount of energy needed to add the 3rd electron is larger that the one for the 2nd electron (additional energy is proportional to side gate voltage difference between $N$ and $N + 1$). In same manner the additional energy for the 7th electron is higher

than for the 6th one and for the 13th higher than for the 12th. This can be explained in terms of 2D electron orbits: If we assume that the potential inside the dot is the 2D (due to the very small height of the dot) circularly symmetrical harmonic potential then the Hamiltonian can be written as:

$$H = \frac{p_x^2}{2m_e} + \frac{p_y^2}{2m_e} + \frac{m_e\omega^2}{2}x^2 + \frac{m_e\omega^2}{2}y^2$$

This is the Hamiltonian of two independent harmonic oscillators. Thus, the energies of the quantum dot are given by two quantum numbers:

$$E(n,m) = \hbar\omega\left(n + \frac{1}{2}\right) + \hbar\omega\left(m + \frac{1}{2}\right)$$

Every $(n, m)$ state can be occupied by two electrons with opposite spins, thus $E(0,0) = \hbar\omega$ can be occupied by two electrons, $E(1,0) = E(0,1) = 2\hbar\omega$ can be occupied by four electrons, $E(1,1) = E(2,0) = E(0,2) = 3\hbar\omega$ can be occupied by six electrons and so on. This results in the fact, that the 3rd, the 7th and the 13th electrons need more additional energy than their predecessors.

This simple theoretical model of a quantum dot yields several results:

- The quantum dot can be considered as a 2D artificial atom, since the electrons are squeezed in a flat region
- Its shell structure yields the quantum energy levels and therefore the energy needed to add more electrons
- The shell structure seems to be the one af a 2-dimensional harmonic potential
- The shell structure can therefore be changed via applied voltages
- Therefore the number of electrons on the dot and its thermal fluctuations can be experimentally controlled

The fact, that quantum dots and rings could be realized experimentally and show the properties of artificial atoms mentioned above lead to the growing interest towards this objects in last few decades, since it is experimentally much easier to study and control single quantum dots than single atoms or molecules.

# 3 Experimental survey

Plenty of experiments on quantum rings and quantum dots have been made in last decades in order to study artificial atoms and the effects of magnetic field on them. The purpose of this section is to introduce two of these experiments in order to demonstrate that the theoretical predictions made by the simple models introduced above can be reproduced experimentally. In the first experiment, made by T.Ihn, A. Fuhrer et. al. and described in their paper "Marvellous things in marvellous rings: energy spectrum, spins and persistent currents" from 2002

(see [5]), the Coulomb blockade effect and the dependence of the transport current through a semiconductor ring are measured. The persistent current in a gold loop is the main topic of the second experiment, that was made by V. Chandrasekhar et. al. back in 1991, see [3].

## 3.1 Coulomb blockade and transport current

T. Ihn et. al. managed to oxidize a ring structure upon a AlGaAs/GaAs surface via an atomic force microscope (AFM) and depleted a 2D electron gas below the oxidized region. Here a picture of the ring, fig. 10.

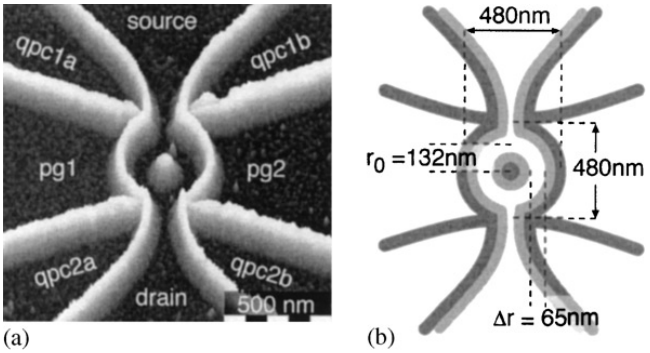



Figure 10: a) AFM picture of the ring, (bright) oxidized regions separate (dark) conductive regions b) Schematic sketch of the ring

The number of electrons on the ring is changed via the voltage applied on the gates pg1 and pg2, which correspond to the side gate, described in section 2.3. The width of the contacts, connecting the ring to the source (drain) can be tuned via the qpc gates. In this experiment a constant source-drain voltage was applied and the dependence of the source-drain current (transport current) on the magnetic field and the side gate voltage was measured. The whole setup was cooled down to $T = 100$ mK. In fig. 11 you can see the results of the measurements.

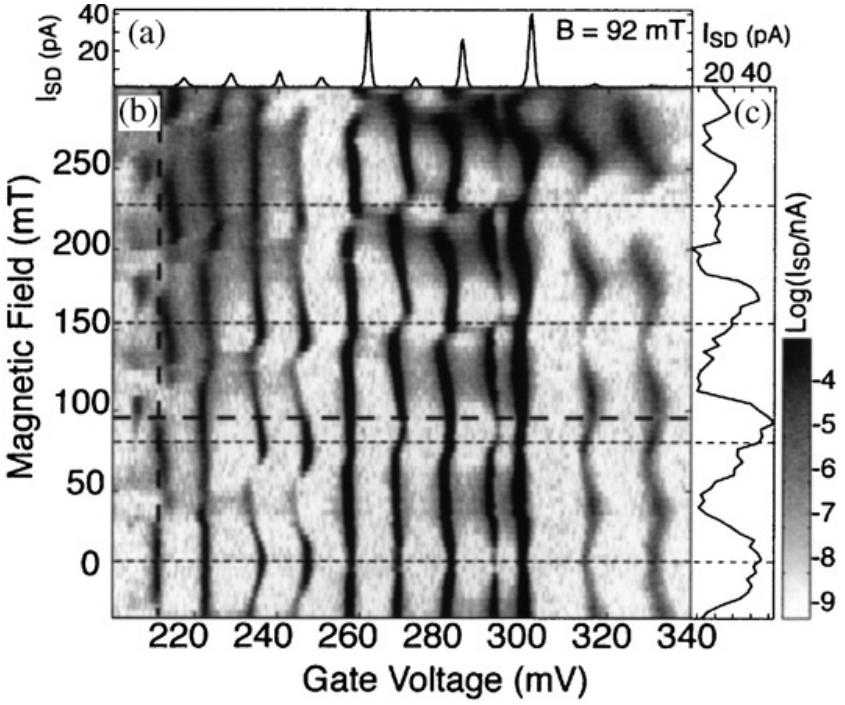



Figure 11: a) Current at constant magnetic field $B = 92$ mT b) Greyscale plot of the source-drain current at constant $V_{SD} = 20$ $\mu$V c) Current at constant side gate voltage $V_G = 200$ mV

Diagram a) shows the dependence of the source-drain current on the side gate voltage. One can clearly see the Coulomb blockade peaks which were explained in section 2.3. More interesting is the diagram c) which shows the current through the ring versus the magnetic field applied on the ring. There is a clear periodic oscillation with the fundamental Aharonov-Bohm period $\Phi_0 = \frac{hc}{e}$ due to interferometry effects which was exactly predicted by the theoretical model in section 2.2.

## 3.2 Persistent current

V.Chandrasekhar et. al. measured the magnetic response of a single, isolated gold loop penetrated by a magnetic flux in 1991. In this section their experiment, which was another evidence for persistent currents is presented in detail.The information is cited from the paper "Magnetic Response of a Single, Isolated Gold Loop", written by V. Chandrasekhar et. al, see [3]. First some technical details:

- Gold loops were fabricated on Si substrates
- The whole sample was cooled down to 300 mK
- Measurements were made on three different gold loops: Two rings with diameters 2.4 and 4.0 $\mu$m and a $1.4\mu$m$\times 2.6\mu$m rectangle

- Loop linewidth of 90 nm and thickness of 60 nm

The phase coherence length $l_\Phi$ was estimated to be 12 $\mu$m and essentially temperature independent below 300 mK, meaning that the electrons kept their phase memory while moving once around the loop with the biggest circumference. Thus, it was possible to observe interferometry effects in these loops. You can see the schematic setup of the experiment in figure 12.

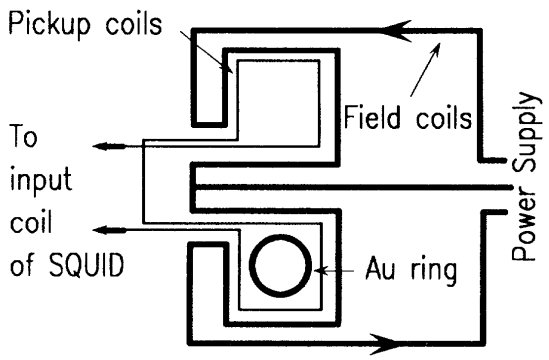



Figure 12: Schematic setup of gold loop measurements. Due to geometry of pickup and field coils the pickup coils only notice the magnetic response of the gold loop

A DC current was swept continuosly through the field coils, generating a magnetic field in the range of $24 - 35$ G (calculated from the geometry of the coils). This current induced a magnetic field penetrating the Au ring and both loops of the pickup coil. Due to the winding of the pickup loops there was no current induced in the pickup coil by the magnetic field of the field coils (currents induced in the upper and in the lower loops of the pickup coil cancel each other). Thus, the only response received by the pickup coils is the one from the magnetic field of the gold ring, induced by persistent currents in the ring. The inductance $L$ between the Au loop and the pickup coils was estimated to be $L \simeq 1$ pH in the used samples. Thus, backcoupling effects between the gold loop and the pickup coils could be neglected. You can see the results obtained from this setup in fig. 13.

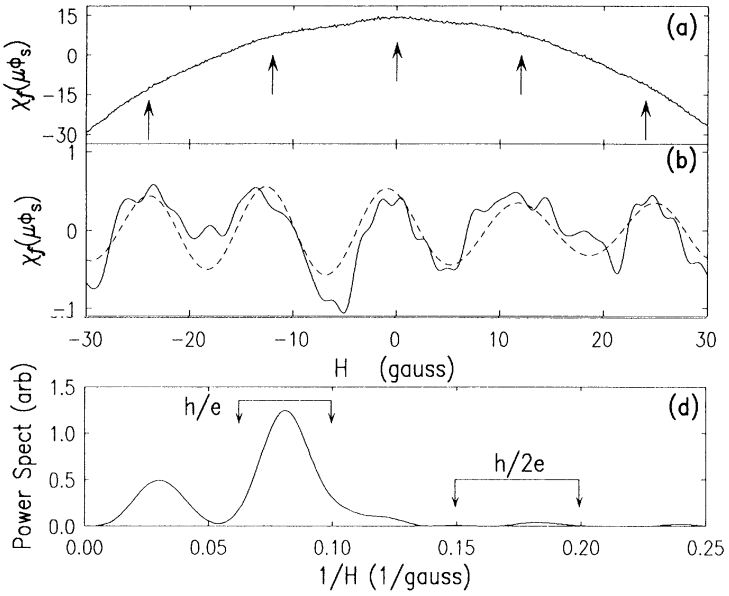



Figure 13: Magnetic response of a gold loop. a) Original signal b) Data of (a) with quadratic background subtracted c) Fourier transform of (b)

From figure 13 (a) one can see, that the signal is, to first order, quadratic in field. This quadratic behavior can be observed even without the gold loop. After subtracting the quadratic background fig. 13 (b) is obtained. Clear periodic oscillations of the magnetic response of the gold loop as a function of the magnetic field can be observed, with a fundamental period corresponding to the flux $\Phi_0$ through the loop, as predicted by the theoretical model of quantum rings with non-interacting particles in section 2.2.

# 4 Summary and outlook

Summarizing the main points of this report one can say that some of the results obtained from experiments on quantum rings agree very well with the theoretical predictions which are based on few assumptions. Here are these assumptions and main results of the theoretical approach:

- Non-interacting spinless fermions on a 1D ring
- Magnetic flux through the ring but field-free particles
- Persistent and transport current periodic in flux $\Phi$, interferometry effects

And the corresponding experimental observations:

- Agreement with the theoretical prediction of periodicity of the current

- Interferometry effects at measurement of transport current through the ring

In addition one should mention that though the periodicity of the currents could be predicted by the theory and measured in experiment, the amplitude of the oscillations obtained from measurements doesn't agree with the theoretical model: The experimentally measured amplitude is at least few orders higher than the predicted one (results from several experiments carried out with different setups).

Next steps in studying quantum rings would be to consider theoretical models which implement mutual particle interaction and interaction of particles with impurities positioned along the circumference of the ring. At present, one such approach is being made by Marc Siegmund from Lehrstuhl für Theoretische Festkörperphysik, Friedrich-Alexander-University Erlangen-Nürnberg during the work on his PhD thesis. He uses density functional theory to calculate the behavior of several interacting electrons on a quantum ring (Wigner crystallization) and their interaction with a single localized impurity.

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