

Approximating the Orthogonal Knapsack Problem for Hypercubes

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Abstract. Given a list of d -dimensional cuboid items with associated profits, the *orthogonal knapsack problem* asks for a packing of a selection with maximal profit into the unit cube. We restrict the items to hypercube shapes and derive a $(\frac{5}{4} + \epsilon)$ -approximation for the two-dimensional case. In a second step we generalize our result to a $(\frac{2^d+1}{2^d} + \epsilon)$ -approximation for d -dimensional packing.

1 Introduction

The *knapsack problem* is one of the most fundamental optimization problems in computer science. The classical one-dimensional variant and its applications are subject to a great number of articles, see [1] and [2] for surveys. Not surprisingly, a geometrical generalization called *d -dimensional orthogonal knapsack problem* (*OKP- d*) is also popular. It is defined as follows.

Given a list $I = (r_1, \dots, r_n)$ of cuboid items $r_i = (a_{i,1}, a_{i,2}, \dots, a_{i,d})$ with associated profit $p_i > 0$ and the unit hypercube $B = [0, 1]^d$ as a bin. The objective is to find a *feasible*, i.e., *orthogonal*, *non-rotational* and *non-overlapping* packing of a selection $I' \subset I$ into B such that the overall packed profit is maximized. An *orthogonal* packing requires that the items are packed parallel to the axis of the bin. Items are *non-overlapping* if their interiors are disjoint. For the two-dimensional case, i.e., packing rectangles into a unit square, the best-known general result is a $(2 + \epsilon)$ -approximation given by Jansen and Zhang [3]. As the difficulty of the problem is increasing drastically with the dimension, only recently a $(7 + \epsilon)$ -approximation for the three-dimensional case was derived by Diedrich et al. [4].

In this paper we restrict the items to *hypercube* shapes (squares instead of rectangles in the two-dimensional case) and investigate how much easier the problem becomes. Note that this restriction is quite popular in the literature ([5], [6], [7]) and yields great potential. Bansal et al. [5] showed for two-dimensional *bin packing*, that even though it is APX-complete in the general case, the restriction to *hypercube bin packing* admits an APTAS. Furthermore, their results hold for higher dimensions as well.

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Our Contribution. Our main result is an approximation algorithm for *square packing*, i.e., *hypercube OKP-2*, with an approximation ratio of $(\frac{5}{4} + \epsilon)$. Moreover, we show that our result can be extended to d -dimensional packing, deriving an $(\frac{2^d+1}{2^d} + \epsilon)$ -approximation. Note that we improve the known approximations for general two- and three-dimensional knapsack packing significantly. Furthermore, we reverse the effect of rising approximation ratios for higher dimensions. In fact, our approximation ratio is improving exponentially with the dimension.

Related Problems. Besides the *orthogonal knapsack problem*, there are two other common generalizations of packing problems. The previously mentioned *d -dimensional orthogonal bin packing problem (OBPP- d)* has the objective of minimizing the total number of unit-size bins in order to pack a list I of cuboid items. The *d -dimensional orthogonal strip packing problem (OSPP- d)*, on the other hand, asks to pack into a strip of bounded basis and unlimited height such as to minimize the total height of the packing.

In 1990 Leung et al. [8] proved the NP-hardness in the strong sense for the special case of determining whether a set of squares can be packed into a bigger square or not. Therefore, already a very special two-dimensional case and all generalizations are strongly NP-hard. In spite of that, the NP-hardness of *hypercube* bin, strip and knapsack packing is still an open problem for $d > 2$.

OSPP-2 admits an asymptotic full polynomial time approximation scheme (AFPTAS, see [9]) for the rotational and non-rotational case, see Jansen and van Stee [10], and Kenyon and Rémila [11]. For *OSPP-3*, Jansen and Solis-Oba [12] gave a $(2 + \epsilon)$ -approximation and Epstein and van Stee studied the *z -oriented* variant, giving a 2,25-approximation [13]. For general *OBPP-2*, the best-known result is a 1,691...-approximation by Caprara et al. [14].

Apart from the general *knapsack packing* results mentioned earlier, *OKP-2* has also been studied in different variants. For the restriction to packing squares into a rectangle in order to maximize the number, Jansen and Zhang gave an AFPTAS [7]. Maximizing the packed area of squares admits a PTAS, as Fishkin et al. showed [6]. In the case, that the rectangles are much smaller than the bin, a better approximation is possible. We refer to this case as *packing with large resources*. Fishkin et al. [15] showed that a solution with weight at least $(1 - \epsilon)$ of the optimum can be found if the side length of the bin differ by at least $1/\epsilon^4$.

An application for the two-dimensional *knapsack problem* is job scheduling with a due date, where the jobs have to be assigned to a consecutive line of processors and the overall profit of accepted jobs has to be maximized. Further applications of packing problems include container loading, VLSI design and advertisement placement, i.e., placing rectangular ads on a given board.

In order to generalize our *square packing* result to higher dimensions, we derived an APTAS for d -dimensional *hypercube strip packing* and a result similar to [15] for *hypercube knapsack packing with large resources*. Both results are motivated by their two-dimensional equivalents in [11] and [15] and thus also stand

for themselves. Due to page limitations we present the proofs in the Appendix.

Presentation of the Paper. We begin with some Preliminaries in Section 2. In Section 3 we describe the $(\frac{5}{4} + \epsilon)$ -algorithm for *hypercube OKP-2*. Before the presentation of the generalization in Section 5 we state our results on *hypercube OSPP-d* and on *knapsack packing with large resources* in Section 4. In Section 6 we conclude our presentation and point out future work.

2 Notations and Preliminaries

Since the items are squares (or hypercubes) throughout the paper, we refer to both, the items and their sizes by a_i . Let I be a set of items. We denote the volume of I by $\text{Vol}(I) = \sum_{i \in I} a_i^2$ ($\sum_{i \in I} a_i^d$ in the d -dimensional setting), the profit of I by $p(I) = \sum_{i \in I} p_i$ and the optimal profit of a packing by $\text{OPT}(I)$.

Bansal et al. showed in [5] how to check the *feasibility*, i.e., whether a given set of items can be packed into the bin, in constant time if the number of items is bounded by a constant. We refer to this method by *constant packing*.

Coffman et al. [16] analysed the *Next Fit Decreasing Height (NFDH)* heuristic for the two-dimensional case. Their work was generalized by Bansal et al. [5] for d -dimensional packing. We will use *NFDH* for packing small items. See [5] for a detailed description of the multidimensional *NFDH*.

Lemma 1. *NFDH*

1. The total wasted (unfilled) volume of a packing P of a set I of items smaller than a constant δ into a cuboid bin $B = (b_1, \dots, b_d)$ with $b_i \leq 1$ by *NFDH* is bounded by $\delta \sum_{i=1}^d b_i \leq \delta d$.
2. Given a set of small, i.e., $a_i \leq \delta$ and therefore $\text{Vol}(a_i) \leq \delta^2$, squares S . If the total volume V of the given space is at least δ and the total wasted volume when packing with *NFDH* is at most δ^2 then we can pack the small items with profit at least $(1 - 2\delta)\text{OPT}(S)$.

Proof. Part 1 is shown in [5]. To see Part 2, let $S' = \text{FracKnap}^+(S, V - 2\delta^2)$, where $\text{FracKnap}^+(S, V - 2\delta^2)$ is the optimal solution of the fractional knapsack instance with volume bound $V - 2\delta^2$. The items S are given by their volume and the optimal solution is derived by the well-known greedy algorithm, including a possible fractional item, and thus possibly exceeding the volume bound. Since the volume of every item is at most δ^2 , we get $\text{Vol}(S') \leq V - \delta^2$. Therefore a packing of S' into the volume is possible. Observe that $p(S') \geq \frac{V - 2\delta^2}{V} \text{OPT}(S) \geq (1 - \frac{2\delta^2}{\delta}) \text{OPT}(S) = (1 - 2\delta) \text{OPT}(S)$. \square

To restrict the number of gaps in a packing, Bansal et al. [5] showed

Lemma 2. Let P be a packing of m hypercubes in $[0, 1]^d$. Then, the remaining space $[0, 1]^d \setminus P$ can be divided into at most $(2m)^d$ non-overlapping cuboids.

For $d = 2$, the number of rectangles is bounded by $3m$.

Finally, we introduce a *shifting technique* that we use several times to free a given line L inside a packing P without losing to much profit. Fishkin et al. used a similar technique in [15].

Lemma 3. *Given a packing P of a list $I = (a_1, \dots, a_n)$ of small ($a_i \leq \delta$) squares into a rectangle $R = (w, h)$ with width $w = 1$ and a vertical line L . If $\delta \leq \frac{1}{2}$, we can derive a packing P' of a selection $I' \subset I$ into R with profit $p(I') \geq (1 - 4\delta)p(I)$ such that L does not intersect with any item.*

Proof. Let I_L be the set of items that intersect L . Partition R into $l = \lfloor \frac{1}{\delta} \rfloor \geq \frac{1}{\delta} - 1$ rectangular slices S_1, \dots, S_l of width δ and a possible smaller one by drawing cutting lines with a distance of δ parallel to the bins height as in Figure 1(a). Find an index i such that the items, that intersect with S_i have minimal profit. Remove all items that intersect with S_i and copy the items I_L left-aligned into S_i . The remaining profit is

$$p(S') \geq p(S) - \frac{2p(S)}{\frac{1}{\delta} - 1} \geq (1 - 4\delta)p(S)$$

since every item intersects with at most two rectangle S_i, S_{i+1} . \square

Note, that the proof is also valid for a rectangle R with $h = 1$ and a horizontal line L . Furthermore, it is not necessary that R is a rectangle as long as the cutting line is at the thinnest part of R so that copying the items is possible, see Figure 1(b) for another possible setting.

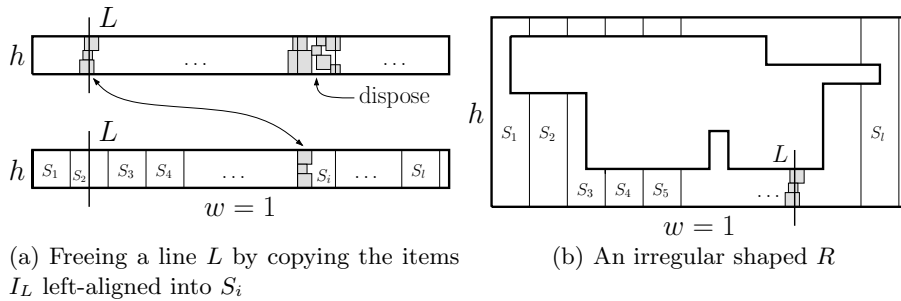


Fig. 1.

3 Square Packing

We now describe our main result for the two-dimensional case. Later we will generalize it for d -dimensional packing. In order to ease the generalization, we

split the description into several parts.

Outline. The first step of the algorithm is a separation of the items into sets of large, medium and small items. This yields a gap in size between large and small items and a profit of the medium items that is negligible. Since the number of large items in the bin is bounded by a constant, we can enumerate over all possible selections and thus assume the knowledge of an optimal packing of large items. After that, we consider three different cases for packing: 1) the large items leave *enough remaining space* to pack the small items, 2) there are *several large items*, and 3) there is only *one very large item*.

We derive almost optimal solutions for the first and third case and an almost $\frac{k+1}{k}$ -optimal solution for the second case, where k is the number of large items. By showing, that any packing with $k < 4$ can be reduced to either the first or the third case, we derive an overall approximation ratio of $(\frac{5}{4} + \epsilon)$.

Let $0 < \epsilon \leq 1/2^{10}$, $\epsilon' = \epsilon/3$. The following *separation technique* divides an optimal solution I_{opt} into sets L_{opt} of large, M_{opt} of medium and S_{opt} of small items such that $p(M_{opt}) \leq \epsilon' \text{OPT}(I)$ and thus we can neglect the medium items.

Separation Technique. Let $r = \lceil 1/\epsilon' \rceil$. Consider an optimal solution I_{opt} and the sequence $\alpha_0 = \epsilon'$, $\alpha_{i+1} = \alpha_i^4 \epsilon'$ for $i = 0, \dots, r$. Define the partition of I_{opt} into sets $M_0 = \{s \in I_{opt} : s \geq \alpha_1\}$, $M_i = \{s \in I_{opt} : s \in [\alpha_{i+1}, \alpha_i]\}$ for $1 \leq i \leq r$ and $M_s = \{s \in I_{opt} : s < \alpha_{r+1}\}$. Observe, that there is an index $i \in \{1, \dots, r\}$ such that $p(M_i) \leq \epsilon' p(I_{opt}) = \epsilon \text{OPT}(I)$. Let $L_{opt} = M_0 \cup \dots \cup M_{i-1}$ be the set of large, $M_{opt} = M_i$ the set of medium and $S_{opt} = M_{i+1} \cup \dots \cup M_r \cup M_s$ the set of small items. Thus $p(L_{opt} \cup S_{opt}) \geq (1 - \epsilon') \text{OPT}(I)$ and it is sufficient to approximate this almost optimal solution. Let $S = \{s \in I : s < \alpha_{i+1}\}$, obviously $S_{opt} \subset S$ and thus $\text{OPT}(L_{opt} \cup S) \geq (1 - \epsilon') \text{OPT}(I)$.

Since $s \geq \alpha_i$ for $s \in L_{opt}$, there are at most $1/\alpha_i^2$ items in L_{opt} . Thus we can enumerate over all $i \in \{1, \dots, r\}$ and L with $|L| \leq 1/\alpha_i^2$ and use the *constant packing* method to check the feasibility of L . Hence assume the knowledge of i and L_{opt} . Let $P_{L_{opt}}$ be a packing of L_{opt} by the constant packing method.

The gap in size between the large and the small items is needed to obtain an efficient packing of some of the small items in S with *NFDH* into the gaps of $P_{L_{opt}}$. Since $|L_{opt}| \leq 1/\alpha_i^2$, there are at most $3/\alpha_i^2$ gaps in $P_{L_{opt}}$ - see Lemma 2. Lemma 1 Part 1 bounds the wasted volume for every gap by $2\alpha_{i+1}$. Hence we can bound the overall wasted volume of a packing with *NFDH* of the small items in S into the gaps of $P_{L_{opt}}$ by $\frac{3}{\alpha_i^2} \cdot 2\alpha_{i+1} = 6 \frac{\alpha_i^4 \epsilon'}{\alpha_i^2} = 6\epsilon' \alpha_i^2 \leq \alpha_i^2$, which is a lower bound for the volume of an item in L_{opt} .

Now let us see, how to derive a packing in three different cases: 1) enough remaining space for the small items ($\text{Vol}(L_{opt}) \leq 1 - \alpha_i$), 2) several large items ($|L_{opt}| = k$), and 3) one very large item ($a_{max} \geq 1 - \epsilon'^4$), where a_{max} is the biggest item in L_{opt} .

Lemma 4. Enough Remaining Space

If $\text{Vol}(L_{opt}) \leq 1 - \alpha_i$, we can find a selection $S' \subset S$ of small items such that L_{opt} and S' can be packed together and $p(L_{opt} \cup S') \geq (1 - 3\epsilon')\text{OPT}(I)$.

Proof. The remaining space is at least α_i and the overall wasted volume is at most α_i^2 . As all small items have size at most $\alpha_{i+1} \leq \alpha_i$ we can apply Lemma 1 Part 2 with $\delta = \alpha_i$ to find a feasible selection $S' \subset S$ with $p(S') \geq (1 - 2\delta)\text{OPT}(S) \geq (1 - 2\epsilon')\text{OPT}(S)$, where $\text{OPT}(S)$ is the optimal profit for packing S into the remaining space. \square

Lemma 5. Several Large Items

If $|L_{opt}| = k$, we can find a selection $S' \subset S$ of small items such that L_{opt} and S' can be packed together and $p(L_{opt} \cup S') \geq (\frac{k}{k+1} - 2\epsilon')\text{OPT}(I)$.

Proof. Let $\text{Knapsack}(S, V, \epsilon)$ denote to a solution with accuracy ϵ for a one-dimensional knapsack instance with items S and volume bound V . The items are given by their volume. Let $S' = \text{Knapsack}(S, 1 - \text{Vol}(L_{opt}), \epsilon')$. Note that $p(L_{opt} \cup S') \geq (1 - 2\epsilon')\text{OPT}(I)$. Consider the packing $P_{L_{opt}}$ and use *NFDH* to add as much as possible of S' into the gaps. Let the profit be P_1 . If S' is completely packed, $P_1 = p(L_{opt} \cup S') \geq (1 - 2\epsilon')\text{OPT}(I)$. Otherwise consider a second packing. Therefore remove the item a^* with lowest profit from L_{opt} and pack the remaining items of L_{opt} together with S' into a bin. This is possible since $\text{Vol}(a^*) \geq \alpha_i^2$ and the total waste is bounded by α_i^2 . Let this profit be P_2 . We state that $\max(P_1, P_2) \geq \frac{k}{k+1}p(L_{opt} \cup S') \geq (\frac{k}{k+1} - 2\epsilon')\text{OPT}(I)$. Assume $L_{opt} = \{a_1, \dots, a_k\}$ and $a^* = a_k$. Then,

$$P_1 \geq \sum_{i=1}^k p_i \geq k p_k \quad \text{and} \quad P_2 = \sum_{i=1}^{k-1} p_i + p(S') = p(L_{opt} \cup S') - p_k$$

For $p_k \in [0, \frac{p(L_{opt} \cup S')}{k+1}]$, $P_2 \geq p(L_{opt} \cup S') - \frac{p(L_{opt} \cup S')}{k+1} \geq \frac{k}{k+1}p(L_{opt} \cup S')$ and for $p_k \in [\frac{p(L_{opt} \cup S')}{k+1}, \frac{p(L_{opt} \cup S')}{k}]$, $P_1 \geq \frac{k}{k+1}p(L_{opt} \cup S')$. Note, that $p_k \leq \frac{p(L_{opt} \cup S')}{k}$ as a_k is the item with lowest profit in L_{opt} . \square

Lemma 6. One Very Large Item

If $a_{max} \geq 1 - \epsilon'^4$, we can find a selection $S' \subset S$ of small items such that L_{opt} and S' can be packed together and $p(L_{opt} \cup S') \geq (1 - 3\epsilon')\text{OPT}(I)$.

Proof. The proof consists of two parts. First we show that the big item a_{max} can be packed into the lower left corner of the bin. Second we use the result for *packing with large resources* by Fishkin et al. [15] to find an almost optimal packing for the remaining space.

Consider an optimal packing of I_{opt} where a_{max} is not placed in the lower left corner. Notice that the free space to all sides has width at most $1 - a_{max} \leq \epsilon'^4$. Draw three lines S_1, S_2, S_3 as on the left side of Figure 2(a). As the items might have high profit we cannot dispose them directly, but with the *shifting technique* of Lemma 3 and $\delta = \epsilon'^4$ we obtain a packing without any item intersecting lines

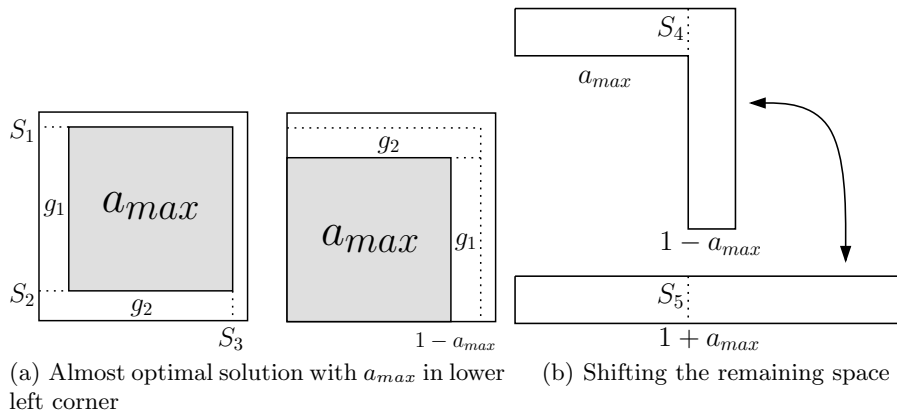


Fig. 2. One very large item

S_1, S_2, S_3 . Thus replace the packing as in Figure 2(a) on the right side, such that a_{max} is placed in the lower left corner.

For $\epsilon \leq 1/2^{10}$, Fishkin et al. [15] described an algorithm that finds a packing for a subset S' of a set of rectangles S into a bin $(1, b)$ where $b \geq 1/\epsilon^4$ with profit $p(S') \geq (1 - \epsilon)\text{OPT}(S)$. We can consider the remaining space in the bin as a strip of size $(1 - a_{max}, 1 + a_{max})$ by cutting at S_4 and rotating a part of the space as shown in Figure 2(b). Scaling this strip and all small items by $\frac{1}{1 - a_{max}}$ gives a strip of size $(1, b)$ where $b = \frac{1 + a_{max}}{1 - a_{max}} \geq 1/\epsilon^4$ (as $a_{max} \geq 1 - \epsilon^4$). Thus we can find a packing with profit at least $(1 - \epsilon')\text{OPT}(S)$. By cutting again at S_5 , the solution can be adopted to the original shape. The rotation is possible since we only have square items. As we have a total of five applications of the *shifting technique*, the loss is bounded by $5 \cdot 4\epsilon^4\text{OPT}(S) \leq \epsilon'\text{OPT}(S)$. \square

We now give a simple but very important lemma, which takes the full advantage of the square shapes of the items, namely that any packing with $k < 4$ large items can be reduced to either the first or the third case. Our intuition is, that it is impossible to fill a unit-size bin with either two or three equally big squares. This also turns out to be the reason for the improving approximation ratio with higher dimensions, e.g., either one very large or more than seven cubes are needed to fill a cube bin almost completely.

Lemma 7. *If $|L_{opt}| < 4$, then either $\text{Vol}(L_{opt}) \leq 1 - \alpha_i$ or $a_{max} \geq 1 - 1/\epsilon^4$.*

Proof. Suppose that $|L_{opt}| \in \{2, 3\}$. If $a_{max} \leq 1/2$, then $\text{Vol}(L_{opt}) \leq 3/4 \leq 1 - \alpha_i$. With $a_{max} > 1/2$ the smaller items in L_{opt} can have a size of at most $1 - a_{max}$ so that L_{opt} is still feasible. As there are at most two more items in L_{opt} , we can bound the total volume by $\text{Vol}(L_{opt}) \leq f(x) \leq x^2 + 2(1 - x)^2$. It is easy to show, that $f(x) \leq 1 - \alpha_1 = 1 - \epsilon'^5$ for $x \in [\frac{1}{2}, 1 - \epsilon'^4]$ - see also Figure 4 for the multidimensional case. \square

The complete algorithm A_2 is summed up in Algorithm 1. The following theorem is immediate since $\frac{1}{\frac{4}{5}-2\epsilon'} \leq \frac{5}{4} + \epsilon$.

Theorem 1. *There is a polynomial time algorithm for hypercube OKP-2 with performance ratio $(\frac{5}{4} + \epsilon)$.*

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for every  $i \in \{1, \dots, r\}$  and feasible  $L \subset \{s \in I : s \geq \alpha_i\}$  with  $|L| \leq 1/\alpha_i^2$  do
  case  $\text{Vol}(L) \leq 1 - \alpha_i$ : solve almost optimal with Lemma 4
  case  $a_{\max} \geq 1 - \epsilon'^4$ : solve almost optimal with Lemma 6
  case  $|L| \geq 4$ : solve with Lemma 5
end
output the packing with the best profit

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Algorithm 1: $(\frac{5}{4} + \epsilon)$ -algorithm A_2 for square packing

4 Useful Tools for Hypercube Packing

In the previous section we used a result on *packing with large resources* to derive the algorithm for the case of one very large item. In order to generalize our algorithm we need a d -dimensional variant of this result. The original two-dimensional algorithm from Fishkin et al. [15] is based on an AFPTAS for *strip packing (OSPP-2)* by Kenyon and Rémila [11]. Similarly, we require an APTAS for *hypercube strip packing* to derive our result on *hypercube knapsack packing with large resources*. See Appendix 7.1 and 7.2 for the proofs of the following theorems.

Hypercube Strip Packing. Let $C \geq 1$ be a bound for the size of the basis. The d -dimensional *hypercube orthogonal strip packing problem (hypercube OSPP- d)* is defined as follows. Given a list $I = (a_1, \dots, a_n)$ of hypercubes $a_i \in (0, 1]$ and a $(d - 1)$ -dimensional cuboid basis of the strip $B = (b_1, b_2, \dots, b_{d-1})$ with $1 \leq b_i \leq C$. The problem is to find a feasible packing P of I into a strip with basis B and unlimited height so that the total height of the packed items is minimized. Using methods from [11] and [5] we derived an algorithm $A_{Strip, \epsilon}$ that holds

Theorem 2. A_{Strip} is an asymptotic polynomial time approximation scheme (APTAS) for hypercube OSPP- d with additive constant $K_{Strip, \epsilon}$ for fixed $\epsilon > 0$ and $C \geq 1$.

Hypercube Knapsack Packing with Large Resources. The *hypercube OKP- d with large resources* is defined as follows. Given a list $I = (a_1, \dots, a_n)$ of hypercubes $a_i \in (0, 1]$, associated profits $p_i > 0$ and a bin $B = (b_1, b_2, \dots, b_d)$ with sizes $b_i \geq 1$. The problem is to find a feasible packing P of a selection $I' \subset I$

into the bin B with maximal profit. Let $V = \prod_{i=1}^d b_i$ be the volume of the bin. Using the algorithm A_{Strip} and ideas from [15], we derived an algorithm A_{LR} that satisfies

Theorem 3. *If $V \geq K_{LR,\epsilon}$ then algorithm A_{LR} finds a feasible packing for a selection $I' \subset I$ with profit at least $(1 - \epsilon)\text{OPT}(I)$.*

The running time of A_{LR} is polynomial and $K_{LR,\epsilon}$ is constant for fixed $\epsilon > 0$.

5 Hypercube Knapsack Packing

Now we are ready to present the generalization of our main result, a $(\frac{2^d+1}{2^d} + \epsilon)$ -approximation for *hypercube knapsack packing*. In the *square packing* algorithm we considered three different cases, packing with *enough remaining space*, packing with *several large items* and packing with *only one large item*. The latter case was motivated by the observation, that three squares cannot fill a unit bin almost completely unless one of the squares is huge. This observation is generalized to a number of $2^d - 1$ hypercubes in the d -dimensional case.

Outline. First, we give new parameters for the separation step such that the first two cases hold even for hypercubes. Second, we show how to handle the third case, applying A_{LR} from the previous section. Finally, we observe that for a number of up to $2^d - 1$ hypercubes, either the remaining space is big enough or there is only one very large item.

Separation. Let $\epsilon' = \epsilon/3$ and $K \geq K_{LR,\epsilon'}$, the constant for algorithm A_{LR} as in the previous section. Let $r = \lceil 1/\epsilon' \rceil$.

Use the sequence $\alpha_0 = \frac{1}{K}, \alpha_{i+1} = \alpha_i^{3d}\epsilon'$ for $i = 0, \dots, r$ to separate an optimal solution I_{opt} into the sets L_{opt}, M_{opt} and S_{opt} as before. Similar to the *square packing* algorithm, the parameters α_i are chosen such that the overall wasted volume of a packing of small items into the gaps of L with *NFDH* is bounded by α_i^d , the lower bound of the volume of a large item in L - see Lemmas 1 and 2. Again we enumerate over all $i \in \{1, \dots, r\}$ and $|L| \leq 1/\alpha_i^d$ and assume the knowledge of i and L_{opt} .

Since the overall wasted volume is bounded by the size of an item in L_{opt} , the first two cases can be handled similarly - see Lemma 4 and Lemma 5.

Now, we show how an almost optimal packing can be derived for $a_{max} \geq 1 - \frac{1}{K}$. First, we show that a special packing structure, similar to packing a_{max} into the lower left corner, does not change the optimal value significantly and second, we use the *shifting technique* and some rotations to apply Theorem 6. Note, that the *shifting technique* is similar for d -dimensional hypercubes, as long as one direction of the space R has length 1.

Well-structured Packing. A packing P is called *well-structured* if the biggest item a_{max} is located in the origin $(0, \dots, 0)$ and the hypercube space

of size $1 - a_{max}$ in the opposite corner as well as all hyperplanes, defined by the facets of a_{max} are completely free of items. See Figure 3(a) for a *well-structured* packing. Similar to the two-dimensional case, we can apply the *shifting technique* to reorder an optimal solution (see Appendix 7.3 for a detailed proof)

Lemma 8. *There is a well-structured packing of a selection $I' \subset I_{opt}$ with profit $p(I') \geq (1 - 2\epsilon')OPT(I)$.*

Applying Algorithm A_{LR} of Theorem 6. We cut and rotate the remaining space of a *well-structured* packing of I' around a_{max} such that it builds a cuboid bin that is much bigger than the remaining items. Then we apply Theorem 6 and by cutting again and reassembling to the original position a valid solution is derived.

Observe, that the remaining space in the bin, with the exception of a hypercube of size $1 - a_{max} \leq \frac{1}{K}$ in the opposite corner of the origin, can be divided into d differently rotated spaces S_1, \dots, S_d of size $(1 - a_{max}, a_{max}, \dots, a_{max}, 1)$ - see Figure 3(b). Note that, since we consider a *well-structured* packing, all items of the near optimal solution I' are completely included in one of these spaces. Rotate all spaces into the same orientation, assemble them to a bin of size $(1 - a_{max}, a_{max}, \dots, a_{max}, d)$ and scale the bin and all small items with $\frac{1}{1 - a_{max}}$. The volume of the bin is bigger than $\frac{1}{1 - a_{max}} \geq K$ (since $a_{max} \geq 1 - \frac{1}{K}$). So we can apply Theorem 6 and therefore find a packing for a selection S' of items with profit $p(L \cup S'') \geq (1 - 3\epsilon')OPT(I')$.

Reassembling the strip-like bin requires $d - 1$ applications of the *shifting technique* and can thus be done with losing at most another $\epsilon'OPT(I)$ of the profit. Let S' be the set of small items after the reassembling. We proved

Lemma 9. *If $a_{max} \geq 1 - \frac{1}{K}$, we can find a selection $S' \subset S$ of small items such that L and S' can be packed together and $p(L \cup S') \geq (1 - 4\epsilon')OPT(I)$.*

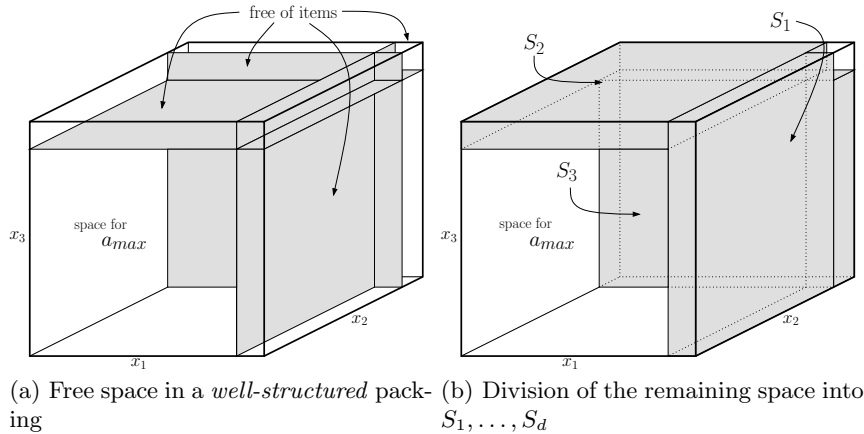


Fig. 3. Packing with one very large item a_{max}

Now let us see that, if $|L_{opt}| < 2^d$, either $\text{Vol}(L_{opt}) \leq 1 - \alpha_i$ or $a_{max} \geq 1 - \frac{1}{K}$. Similar to the two-dimensional analysis, we get a volume bound of $\text{Vol}(L_{opt}) \leq f_d(x) = x^d + (2^d - 1)(1 - x)^d$ for $x \in [\frac{1}{2}, 1]$, see Figure 4. With the second derivate it is easy to see, that $f_d(x) \leq 1 - \frac{1}{2^d}$ for $x \in [\frac{1}{2}, \frac{3}{4}]$ and $f_d(x) \leq x$ for $x \in [\frac{3}{4}, 1]$. Thus $a_{max} \leq 1 - \frac{1}{K}$ implies $\text{Vol}(L_{opt}) \leq 1 - \frac{1}{K}$ for $\frac{1}{K} \leq \frac{1}{2^d}$. Note that $\frac{1}{K} \leq \frac{1}{2^d}$ can be achieved by choosing $K \geq 2^d$. See Appendix 7.4 for more details. We showed

Theorem 4. *There is a polynomial time algorithm for hypercube OKP-d with performance ratio $(\frac{2^d+1}{2^d} + \epsilon)$.*

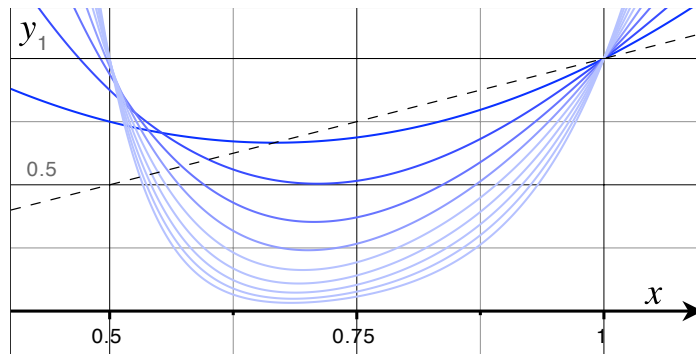


Fig. 4. The volume functions $f_d(x)$ for $d = 2, \dots, 10$ and $x \in [\frac{1}{2}, 1]$ (solid), and the, on $x \in [\frac{3}{4}, 1]$, dominating function $g(x) = x$ (slashed)

6 Conclusion and Future Work

For the special case of packing hypercube items we derived an approximation algorithm for $OKP-d$ with performance ratio $(\frac{2^d+1}{2^d} + \epsilon)$ that is, surprisingly, improving with the dimension. Already for the two- and three-dimensional case, we significantly improve upon the best-known general algorithms.

We gave PTAS-like approximations for the cases that either the remaining volume after packing the large items is big enough or there is only one very large item. In the case of *several large items*, the gap structure becomes more complicated with increasing number of items. Although for *square packing* it seems to be possible to handle the cases $|L_{opt}| = 4$, since the remaining space has the shape of four strip-like bins, and $|L_{opt}| = 5$, since it can be reduced to either the case with four large items or the case with enough remaining space, we could not derive a general method to cope with large numbers of items in L_{opt} . Further research should thus be concentrated on the case of several large items in order to solve the question whether or not a PTAS for *hypercube OKP-d* exists.

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7 Appendix

In the Appendix we present the proofs that were omitted due to page limitations. Our results on *hypercube OSSP-d* and *hypercube OKP-d with large resources* are technical generalizations of the original results in [11] and [15], respectively. Since we cannot present our result in them same detail as done in the original papers, they are an important reference.

7.1 Hypercube Strip Packing

Here we outline an APTAS for the *hypercube OSPP-d*. The methods for this result are closely related to the work of Bansal et al. [5] and Kenyon, Rémila [11]. Let $\epsilon > 0$ be a given accuracy and C a given constant bound for the side length of the basis. We define *hypercube OSPP-d* as follows.

Given a list $I = (a_1, \dots, a_n)$ of hypercubes $a_i \in (0, 1]$ and a $(d - 1)$ -dimensional cuboid basis $B = (b_1, b_2, \dots, b_{d-1})$ with $1 \leq b_i \leq C$. The problem is to find a feasible packing P of I into a strip with basis B and unlimited height so that the total height of the packed items is minimized.

Outline. First, we use the *separation technique* to derive sets L of large, M of medium and S of small items such that the medium items make up only a marginal amount of the total volume, but the gap in size between the large and the small items is considerable. Second, we group the large items and round them to certain threshold items which are packed almost optimally into the strip. Third, we pack some of the small items into the gaps of the large items and finally, we pack the remaining small items and the medium items separately with *NFDH*.

Preliminaries. We denote the height of a packing of a set I of items by $h(I)$ and the height of an optimal packing by $\text{OPT}(I)$. For a given basis $B = (b_1, \dots, b_{d-1})$ or bin $\tilde{B} = (\tilde{b}_1, \dots, \tilde{b}_d)$ we define the volume and the sum of the surface area of the different facets of B and \tilde{B} by

$$\begin{aligned}
 B_{vol} &= \prod_{i=1}^{d-1} b_i \leq C^{d-1} & B_F &= \sum_{\substack{S \subset \{1, \dots, d-1\} \\ |S|=d-2}} \prod_{i \in S} b_i \leq (d-1)C^{d-2} \\
 \tilde{B}_{vol} &= \prod_{i=1}^d \tilde{b}_i \leq C^d & \tilde{B}_F &= \sum_{\substack{S \subset \{1, \dots, d\} \\ |S|=d-1}} \prod_{i \in S} \tilde{b}_i \leq dC^{d-1}
 \end{aligned}$$

Note that $\tilde{B}_F = B_F + B_{vol}$ for $\tilde{b}_i = b_i$ for $1 \leq i \leq d - 1$ and $\tilde{b}_d = 1$.

The following lemma is proven similar to Lemma 1 Part 1 as in [5].

Lemma 10. *The total wasted (unfilled) volume of a packing P of a set I of items smaller than a constant δ into a bin $\tilde{B} = (\tilde{b}_1, \dots, \tilde{b}_d)$ by NFDH is bounded by δB_F .*

Thus the total wasted volume in a strip of basis B and height 1 is bounded by $\delta(B_F + B_{vol})$.

We introduce a usual analysis for a layer based strip packing, as proven for the three-dimensional case by Miyazawa and Wakabayashi in [17].

Lemma 11. *Let P be a packing of a list I of hypercube items into a strip of basis $B = (b_1, b_2, \dots, b_{d-1})$ such that the items are packed into layers l_1, \dots, l_k and items in layer l_i are not smaller than items in layer l_j for $i < j$. If the area guarantee, i.e. the filled basis of each layer, for all but the last layer is at least s then the height of the strip is bounded by*

$$h(I) \leq \frac{Vol(I)}{s} + a_{max}$$

where a_{max} is the biggest item in I .

Separation. Let $\epsilon' = \epsilon/4$ and $r = \lceil 1/\epsilon' \rceil$. Separate the items I as in the description of the *separation technique* in Section 3, using the sequence $\alpha_0 = \min(\frac{B_{vol}}{B_F} \epsilon', \epsilon')$, $\alpha_{i+1} = \alpha_i^{2d} \frac{\epsilon'}{6^d B_{vol}^{d-1} (B_F + B_{vol})}$ for $i = 1, \dots, r+1$. Note, that we separate I instead of I_{opt} , which does not exist for strip packing. We obtain, that there is an index i and sets L , M and S with $Vol(M) \leq \epsilon' Vol(I)$. By enumerating over all indices $i \in \{1, \dots, r\}$ we can assume the knowledge of a suitable i and therefore L , M and S .

Packing the Large Items. To pack the large items almost optimal, we use a rounding procedure which does not increase the optimal packing significantly but reduces the distinct item sizes to a constant.

All items in L are bigger than α_i . Let n be the number of items in L . Sort L in non-increasing order, partition L into $K = \lceil B_{vol}/(\epsilon' \alpha_i^d) \rceil$ consecutive groups g_1, \dots, g_K of at most $H = \lceil n/K \rceil$ items and round each item to the biggest in the group to derive a rounded set of items J . Bansal et al. showed in [5], that

$$OPT(J) \leq \lceil (1 + \epsilon') OPT(L) \rceil.$$

As the strips basis is bounded and therefore at most $M = \lfloor B_{vol}/\alpha_i^{d-1} \rfloor$ projections of items fit into it, we can find all possible packing patterns of the $(d-1)$ -dimensional projection of the hypercubes with the *constant packing method*. With a linear program as in [11] we can find an optimal *fractional* solution with at most K layers that correspond to a packing pattern. Kenyon and Rémila [11] show how to derive a valid integer packing P_L of L with height $h(L) \leq OPT(J) + K \leq \lceil (1 + \epsilon') OPT(L) \rceil + K$.

Packing the Small Items. In order to pack some of the small items of S together with a packing P_L of L we use *NFDH* in the gaps of P_L . Recall, that there are at most $(2m)^d$ gaps for m items - Lemma 2. We show, that the overall waste after packing the small items is low by counting the free gaps in layers of height 1 and estimating the waste per gap.

Cut the strip at integer heights and examine a layer l_i between two such cuts. Obviously, gaps that intersect with a cut are also cut and thus counted twice. The advantage of cutting the strip into layers is, that the number of large items in each layer is bounded. We add the volume of the adjoining layers l_{i-1} and l_{i+1} into the upper bound for the number of gaps since each item might be included in up to two layers. We can bound the number of items in layer l_i by $3B_{vol}/\alpha_i^d$ and therefore the number of gaps by $6^d B_{vol}^d / \alpha_i^{2d}$.

With Lemma 10, we bound the waste in each gap by $\alpha_{i+1}(B_F + B_{vol})$. Hence the total wasted volume per layer is bounded by

$$\frac{6^d B_{vol}^d}{\alpha_i^{2d}} \cdot \alpha_{i+1}(B_F + B_{vol}) \leq \frac{6^d B_{vol}^d}{\alpha_i^{2d}} \cdot \frac{\alpha_i^{2d} \cdot \epsilon'}{6^d B_{vol}^{d-1}(B_F + B_{vol})} (B_F + B_{vol}) \leq \epsilon' B_{vol}.$$

As we considered a random layer of height 1, this holds for every layer. Thus the total waste in the strip is bounded by $\epsilon' B_{vol} h(L)$ if there are enough items in S . Let S'' be the set of packed and S' the set of remaining small items. Pack S' with *NFDH* in a separate strip P_S .

Packing the Medium Items. In order to pack the medium items M we use the *NFDH* algorithm. Let the derived packing be P_M and its height $h(M)$. Since all items are smaller than $\alpha_i \leq \alpha_0 = \min(\frac{B_{vol}}{B_F} \epsilon', \epsilon')$ and thus (Lemma 10) the area guarantee is at least $(B_{vol} - B_{vol} \epsilon') = (1 - \epsilon') B_{vol}$, we get (see Lemma 11)

$$h(M) \leq \frac{1}{1 - \epsilon'} \frac{Vol(M)}{B_{vol}} + \epsilon' \leq 2\epsilon' OPT(I) + \epsilon'$$

The last inequality is due to $\frac{1}{1 - \epsilon'} \leq 2$ for $\epsilon' \leq 1/2$ and $\frac{Vol(M)}{B_{vol}} \leq \frac{\epsilon' Vol(I)}{B_{vol}} \leq \epsilon' OPT(I)$.

Analysis. The complete algorithm A_{Strip} is given in Algorithm 2. We distinguish between two cases, according to S' after step 3.

First case: $S' = \emptyset$. We showed, that the total height of the strip P_L is $h(L) \leq h(J) \leq [(1 + \epsilon') OPT(L)] + K$. Together with the strip P_M we derived a total height of $h(L) + h(M) \leq [(1 + 3\epsilon') OPT(I)] + K + \epsilon' \leq (1 + 3\epsilon') OPT(I) + K + 2$.

Second case: $S' \neq \emptyset$. In this case, the small items fill up almost all volume of the strip of L . As $Vol(S'' \cup L) \geq (1 - \epsilon') B_{vol} h(L)$ we can estimate the height of $P_{S'}$ (similar to P_M) and P_L by

$$\begin{aligned} h(S \cup L) = h(L) + h(S') &\leq \frac{1}{1 - \epsilon'} \frac{Vol(S'' \cup L)}{B_{vol}} + \frac{1}{1 - \epsilon'} \frac{Vol(S')}{B_{vol}} + \alpha_i \\ &\leq (1 + 2\epsilon') OPT(I) + \epsilon' \end{aligned}$$

as $\frac{Vol(I)}{B_{vol}} \leq \text{OPT}(I)$ and $\frac{1}{1-\epsilon'} \leq 1 + 2\epsilon'$ for $\epsilon' \leq 1/2$. Thus $h(S \cup L) + h(M) \leq (1 + 4\epsilon')\text{OPT}(I) + 2\epsilon'$.

Since $\epsilon' = \epsilon/4$ we get an overall height of at most $(1 + \epsilon)\text{OPT}(I) + K + 2$. We denote the upper bound for the additive constant $K + 2$ by $K_{\epsilon,C} = \lceil 4C^{d-1}/(\epsilon\alpha_{\lceil 4/\epsilon}^d) \rceil + 2$ so that we can refer to it later. We showed

Theorem 5. *The algorithm A_{Strip} is a polynomial time approximation scheme for hypercube OSSP-d with additive constant $K_{\epsilon,C}$. Moreover, the packing of A_{Strip} consists of at most $K_{\epsilon,C}$ layers that are associated to certain packing patterns or to the medium items.*

1. Find $i \in \{1, \dots, r\}$ such that $Vol(M) \leq \epsilon' Vol(I)$,
2. round the items in L and find an almost optimal packing P_L of L ,
3. pack S with *NFDH* into the free space of P_L , denote $S' \subset S$ to the items that could not be packed,
4. pack the remaining items of S' with *NFDH* into a free strip P_S ,
5. pack M with *NFDH* into a free strip P_M ,
6. combine the strips P_L, P_S and P_M to P_I ,
7. output P_I .

Algorithm 2: Algorithm A_{Strip} for hypercube *OSPP-d*

7.2 Hypercube Knapsack Packing with Large Resources

Our second result is the application of the algorithm A_{Strip} on *knapsack packing with large resources* similar to [15], where Fishkin et al. applied an AFPTAS for strip packing [11] on 2-dimensional knapsack packing with large resources.

Definition of Hypercube OKP-d with Large Resources. Let $\epsilon > 0$ be a given accuracy. Let $\epsilon' = \frac{\epsilon}{16d-6}$ and $K_{\epsilon'} = K_{\epsilon', 1/\epsilon'}$ where $K_{\epsilon', 1/\epsilon'}$ is defined as in the previous section with $C = 1/\epsilon'$.

Given a list $I = (a_1, \dots, a_n)$ of hypercubes $a_i \in (0, 1]$, associated profits $p_i > 0$ and a bin $B = (b_1, b_2, \dots, b_d)$ with sizes $b_i \geq 1$ and volume $B_{vol} = \prod_{i=1}^d b_i \geq \frac{K_{\epsilon'}}{\epsilon'^{d+1}}$. The problem is to find a feasible packing P of a selection $I' \subset I$ into the bin B with maximal profit.

Outline. First, cut the bin into several parts and assemble them so that it resembles a strip with bounded basis. This is important to apply the algorithm A_{Strip} of the previous section. Second, we show that there is an “almost” optimal solution (i.e. optimal profit but exceeding the bins size) that has a special structure, resembling the packing of the algorithm of the previous section. Third, we guess this structure by enumerating over all possibilities and use knapsack

instances to select items for the packing. Finally, we pack the selected items into the strip, using the algorithm A_{Strip} and possibly use the *shifting technique* to decrease the height of the packing to fit into the bin.

Assembling a Strip-Like Bin. Let k be the lowest index with $b_k \geq 1/\epsilon'$. Let t be the lowest integer such that $b_k \leq t/\epsilon'$. Cut the bin in direction k into t parts of equal size. Let $b'_k = b_k/t$ be the new size in dimension k . Note that $1/2\epsilon' \leq b'_k \leq 1/\epsilon'$ and therefore the *shifting technique* can be applied with $\delta = 2\epsilon'$, $a_i \leq 1$ and $h \geq 1/\delta$ to remove each cut into the prior part of the bin, losing not more than $8\epsilon'OPT(I)$ (see Lemma 3). Rearrange the bin such that the equal parts are piled up in direction of the last dimension and proceed similarly with the other dimension.

Let all but the last dimension be bounded by $1/\epsilon'$. Thus, $b_d \geq B_{vol}/(\frac{1}{\epsilon'})^{d-1} \geq \frac{K_{\epsilon'}}{\epsilon'^2}$ as the total volume does not change with the rescaling. The total loss in profit is bounded by $8(d-2)\epsilon'OPT(I)$. Suppose a packing P of $I' \subset I$ for the newly assembled bin is known. With reverse engineering we reassemble the original bin and thus derive a valid packing P' for it. The loss is bounded by $8(d-2)\epsilon'p(I')$ again. Thus we approximate the newly assembled bin and refer to this bin from now on. The optimal profit for the new bin is denoted by $OPT_{new}(I) \geq (1 - 16(d-2)\epsilon')OPT(I)$.

Almost Optimal Packing with Special Structure. Consider an optimal solution I_{opt} and the packing P_{Strip} of I_{opt} produced by the strip packing algorithm A_{Strip} of the previous section. The height of the packing is

$$\begin{aligned} h_{Strip}(I_{opt}) &\leq (1 + \epsilon')OPT_{Strip}(I_{opt}) + K_{\epsilon'} \\ &\leq (1 + \epsilon')b_d + \epsilon'^2 b_d \\ &\leq (1 + 2\epsilon')b_d \end{aligned}$$

as $b_d \geq \frac{K_{\epsilon'}}{\epsilon'^2}$ and $OPT_{Strip}(I_{opt}) \leq b_d$. Moreover there are at most $K_{\epsilon'}$ layers that correspond to certain packing patterns or to the medium items.

Since we obviously do not know I_{opt} , we have to guess the structure of P_{opt} . For that purpose, round each layer to the next value in $R = \{(\frac{t\epsilon'}{K_{\epsilon'}}) \cdot b_d | t = 1, \dots, \lceil \frac{K_{\epsilon'}}{\epsilon'} \rceil\}$. Each layer is increased by at most $\frac{\epsilon'}{K_{\epsilon'}} b_d$ and thus the new total height is bounded by $H \leq (1 + 2\epsilon')b_d + K_{\epsilon'} \frac{\epsilon'}{K_{\epsilon'}} b_d \leq (1 + 3\epsilon')b_d$.

Recapitulate, that there is a packing \tilde{P} of I_{opt} with height $\tilde{h} \leq (1 + 3\epsilon')b_d$, optimal profit (for the newly assembled bin), at most $K_{\epsilon'}$ layers with heights $l_i \in R$ and an index i separating the sets L of large, M of medium and S of small items as in the strip packing algorithm.

Enumeration. Enumerate over all possible indices $i \in \{1, \dots, \lceil 1/\epsilon' \rceil\}$, numbers $k \leq K_{\epsilon'}$ of threshold items $t_1 \geq \dots \geq t_k$ from L . Calculate all packing patterns (including an empty one) with these threshold items and enumerate over all

selections of $m \leq k + 1$ packing patterns T_1, \dots, T_m with $T_i = (T_{i1}, \dots, T_{ik})$ and their layer heights $l_1, \dots, l_m \in R$. Let l_m be the layer height for the medium items.

Selecting Items. Assuming that the threshold items $t_1 \geq \dots \geq t_k \in L$, the packing patterns T_1, \dots, T_m with $T_i = (T_{i1}, \dots, T_{ik})$ and their heights $l_1, \dots, l_m \in R$ are known, we use knapsack instances to find almost optimal selections of items of every size.

Let $L_i = \{i \in L : a_i \in [t_i, t_{i+1}]\}$ be the set of items between the threshold sizes (assume that t_{k+1} denotes the smallest item in L). Observe, that in \tilde{P} the total height of slots for L_i is given by $h_i = \sum_{j=1}^{m-1} T_{ji} l_j$. Therefore $L_i \cap I_{opt}$ satisfies $\sum_{i \in L_i \cap I_{opt}} a_i \leq h_i$. Now let $L'_i = \text{Knapsack}(L_i, h_i, \epsilon')$ be our approximation for $L_i \cap I_{opt}$. Let $L' = L'_1 \cup \dots \cup L'_k$.

Analogously, we get a volume bound for $S \cap I_{opt}$ with $V_S = \sum_{j=1}^{m-1} F_j l_j$ where F_j is the free basis in configuration T_i . Moreover we have the volume bound for $M \cap I_{opt}$ as the volume of the layer for the medium items $V_M = l_m B_{vol}$. Define $S' = \text{Knapsack}(S, V_S, \epsilon')$ and $M' = \text{Knapsack}(M, V_M, \epsilon')$ where the items in S and M are given by their volume. The selection of the items is done this way to yield

Lemma 12. *The overall profit of this selection is*

$$p(L') \geq (1 - \epsilon')p(L_{opt})$$

$$p(M') \geq (1 - \epsilon')p(M_{opt})$$

$$p(S') \geq (1 - \epsilon')p(S_{opt})$$

and thus $p(L') + p(M') + p(S') \geq (1 - \epsilon')\text{OPT}_{new}(I)$. Furthermore, the height of a packing P with A_{Strip} , where L_{opt} , M_{opt} and S_{opt} are substituted by L' , M' and S' , respectively, is at most $h \leq \bar{h} + K_{\epsilon'} \leq (1 + 6\epsilon')b_d$

Proof. See Fishkin et al. [15] for a similar proof that can easily be adopted.

Packing L' , M' and S' . Pack $I' = L' \cup M' \cup S'$ with A_{Strip} to derive a packing of height at most $(1 + 6\epsilon')b_d$. Note that the *shifting technique* can also be used in the case that the items have size $a_i \in (0, 1]$ and the height is $h \geq 1/\delta$. Thus apply the *shifting technique* with $\delta = 1/b_d$ to lower the packing under the height of b_d - see also the Note on the *shifting technique* in Section 7.3 - and derive a profit of

$$p(I'') \geq (1 - 4(6\epsilon'b_d + 2)\frac{1}{b_d})p(I') \geq (1 - 25\epsilon')p(I')$$

since $\frac{2}{b_d} \leq \epsilon'$.

```

Assemble a new bin with basis  $B'$ ,
for every  $i \in \{1, \dots, r\}$ ,  $k \leq K_{\epsilon'}$  threshold items  $t_1 \geq \dots \geq t_k$ ,  $m \leq k + 1$ 
packing patterns  $T_1, \dots, T_m$  and their layer heights  $l_1, \dots, l_m \in R$  do
    calculate  $L'_j$ ,  $M'$  and  $S'$  for  $j = 1, \dots, k$ ,
    pack  $L'_j$ ,  $M'$  and  $S'$  with  $A_{Strip}$  into a strip of basis  $B'$ ,
    if height  $h \leq (1 + 6\epsilon')b_d$  then
        keep the solution
    end
end
select the solution with the best profit,
reduce the height with the shifting technique under  $b_d$ ,
cut the bin and reassemble it to the original shape

```

Algorithm 3: Algorithm A_{LR} for hypercube OKP-d with large resources

Algorithm. The overall algorithm A_{LR} is summarized in Algorithm 3. The achieved profit is

$$\begin{aligned}
p(I'') &\geq (1 - 25\epsilon')p(I') \geq (1 - 25\epsilon')(1 - \epsilon')\text{OPT}_{new}(I) \\
&\geq (1 - 26\epsilon')(1 - 16(d - 2)\epsilon')\text{OPT}(I) \\
&\geq (1 - (16d - 6)\epsilon')\text{OPT}(I) \\
&\geq (1 - \epsilon)\text{OPT}(I)
\end{aligned}$$

With $K_{LR,\epsilon} = \frac{K_{\epsilon'}}{\epsilon^d}$ we showed

Theorem 6. A_{LR} is a polynomial time algorithm for hypercube OKP-d with large resources with performance ratio $(1 + \epsilon)$ if the bins size is at least $K_{LR,\epsilon}$.

7.3 Existence of a Well-Structured Packing with Sufficient Profit

We show Lemma 8:

“There is a well-structured packing of a selection $I' \subset I_{opt}$ with profit $p(I') \geq (1 - 2\epsilon')\text{OPT}(I)$.”

Proof. Let (x_1, \dots, x_d) be the position of a_{max} in a packing P_{opt} of an optimal solution $I_{opt} = L_{opt} \cup M_{opt} \cup S_{opt}$. As usual we neglect M_{opt} . Thus $p(L_{opt} \cup S_{opt}) \geq (1 - \epsilon')\text{OPT}(I)$. Let i be an index with $x_i > 0$. We apply the *shifting technique* $2(d - 1)$ times to be able to exchange a_{max} with another part of the bin such that $x'_j = x_j$ for $j \neq i$ and $x_i = 0$.

To do this, free the hyperplanes H_1, \dots, H_{d-1} and $\tilde{H}_1, \dots, \tilde{H}_{d-1}$ that are defined by the facets of a_{max} with the exception of the hyperplanes orthogonal to the i th-dimension - see Figure 5. No item in the hypercuboid C between the lower x_i facet of a_{max} and the bin intersects any of the hyperplanes after this step. Thus a_{max} and C can be exchanged without losing any further profit.

Iterative application of this step gives a sequence of at most $2(d - 1) \cdot d$ applications of the *shifting technique*. Now we free all hyperplanes, that are

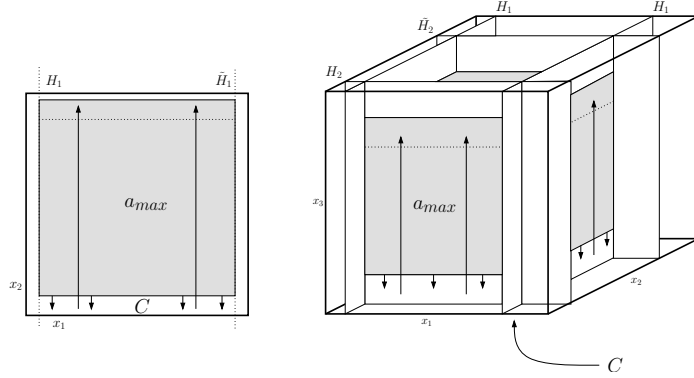


Fig. 5. The hyperplanes H_1, \dots, H_{d-1} and $\tilde{H}_1, \dots, \tilde{H}_{d-1}$ for the application of the *shifting technique* to move a_{max} along x_d for $d = 2$ and $d = 3$

defined by the facets of a_{max} with d applications of the *shifting technique*. Note, that a the *shifting technique* can be extended to free small spaces of height $\leq k\delta$ instead of hyperplanes, losing not more than $4(k+2)\delta$ of the profit. Thus we finally free the hypercube of size $1 - a_{max}$ in the opposite corner of the bin with a final shifting step. In total we need $2(d-1)d + d \leq 2d^2$ application of the *shifting technique* for a hyperplane and one application for a space of height $1 - a_{max}$. Since $a_{max} \geq 1 - \frac{1}{K}$ we can use $\delta = \frac{1}{K}$ and $k = 1$ to get a profit of

$$\begin{aligned} p(I') &\geq p(L_{opt} \cup S_{opt}) - 8d^2 \frac{1}{K} p(L_{opt} \cup S_{opt}) - 4(1+2) \frac{1}{K} p(L_{opt} \cup S_{opt}) \\ &\geq (1 - (8d^2 + 12) \frac{1}{K}) p(L_{opt} \cup S_{opt}) \\ &\geq (1 - \epsilon') p(L_{opt} \cup S_{opt}) \\ &\geq (1 - 2\epsilon') \text{OPT}(I) \end{aligned}$$

for $\epsilon' \leq \frac{1}{8d^2 + 12}$.

Thus we derived a *well-structured* packing P of a selection $I' \subset I_{opt}$ with sufficient profit. \square

7.4 Distinction into Three Cases for *Hypercube OKP-d*:

Assume that a_{max} is the biggest item in L_{opt} and $|L_{opt}| < 2^d$. We show that $a_{max} \leq 1 - \frac{1}{K}$ implies $\text{Vol}(L_{opt}) \leq 1 - \frac{1}{K} \leq 1 - \alpha_i$.

We distinguish the cases 1) $a_{max} \leq \frac{1}{2}$, 2) $a_{max} \in [\frac{1}{2}, \frac{3}{4}]$ and 3) $a_{max} \in [\frac{3}{4}, 1]$.

If $a_{max} \leq \frac{1}{2}$ we get $L_{opt} \leq (2^d - 1)(\frac{1}{2})^d = 1 - (\frac{1}{2})^d \leq 1 - \frac{1}{K}$. Now we consider the function $f_d(x) = x^d + (2^d - 2)(1 - x)^d$ that is an upper bound for the volume - see Figure 4. Consider the two variants $g_d(x) = f_d(x) - (1 - (\frac{1}{2})^d)$ on the domain $x \in [\frac{1}{2}, \frac{3}{4}]$ and $h_d(x) = f_d(x) - x$ on $x \in [\frac{3}{4}, 1]$. Note that the second derivative of

all three functions is equal, e.g. $f_d(x)'' = (d-1)dx^{d-2} + (d-1)d(2^d-2)(1-x)^{d-2}$ and greater than zero on $x \in [\frac{1}{2}, 1]$, since both summands are positive on this domain. Thus f_d , g_d and h_d are convex on $x \in [\frac{1}{2}, 1]$ and g_d and h_d have their maximum in $\frac{1}{2}$ or $\frac{3}{4}$, and $\frac{3}{4}$ or 1, respectively.

Claim. $f_d(\frac{3}{4}) \leq f_{d-1}(\frac{3}{4})$ for $d \geq 3$.

Proof.

$$\begin{aligned}
& f_d\left(\frac{3}{4}\right) \leq f_{d-1}\left(\frac{3}{4}\right) \\
\Leftrightarrow & \left(\frac{3}{4}\right)^d + (2^d - 2) \left(\frac{1}{4}\right)^d \leq \left(\frac{3}{4}\right)^{d-1} + (2^{d-1} - 2) \left(\frac{1}{4}\right)^{d-1} \\
& \Leftrightarrow \frac{3^d + 2^d - 2}{4^d} \leq \frac{3^{d-1} + 2^{d-1} - 2}{4^{d-1}} \\
& \Leftrightarrow 3^d + 2^d - 2 \leq (3^{d-1} + 2^{d-1} - 2)4 \\
& \Leftrightarrow 3^d + 2^d - 2 \leq 3 \cdot 3^{d-1} + 3^{d-1} + 2 \cdot 2^{d-1} + 2 \cdot 2^{d-1} - 8 \\
& \Leftrightarrow 3^d + 2^d - 2 \leq 3^d + 2^d + 2^d + 3^{d-1} - 8
\end{aligned}$$

and this is true since $d \geq 3$ and thus $2^d \geq 8$ and $3^{d-1} \geq 9$.

The evaluation at the points of interest gives $g_d(\frac{1}{2}) = 0$ and $g_d(\frac{3}{4}) = f_d(\frac{3}{4}) - (1 - (\frac{1}{2})^d) \leq f_2(\frac{3}{4}) - (1 - (\frac{1}{2})^d) \leq f_2(\frac{3}{4}) - (1 - (\frac{1}{2})^2) = g_2(\frac{3}{4}) = -\frac{1}{16}$. Thus $g_d(x) \leq 0$ and $f_d(x) \leq 1 - (\frac{1}{2})^d$ for $x \in [\frac{1}{2}, \frac{3}{4}]$.

For h_d we get $h_d(\frac{3}{4}) = f_d(\frac{3}{4}) - \frac{3}{4} \leq f_2(\frac{3}{4}) - \frac{3}{4} = -\frac{1}{16}$ and $h_d(1) = 0$. Thus $h_d(x) \leq 0$ and $f_d(x) \leq x$ for $x \in [\frac{3}{4}, 1]$. We showed, that $x \leq 1 - \frac{1}{K}$ implies $f_d(x) \leq 1 - \frac{1}{K}$.