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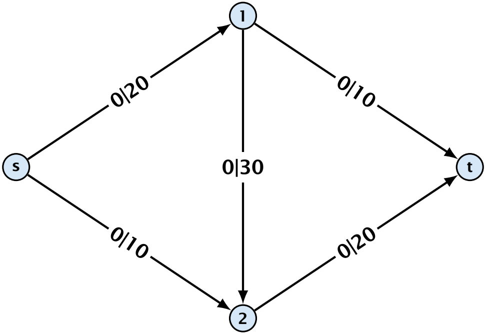

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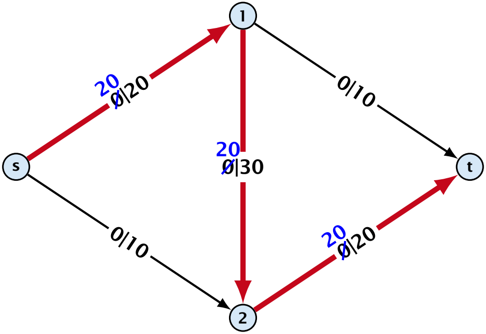

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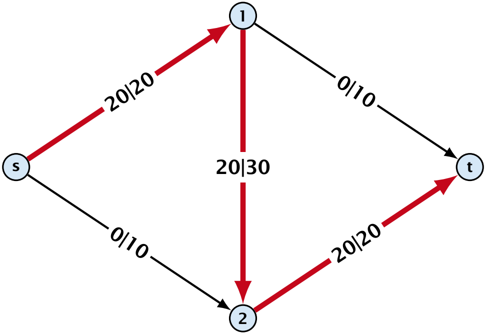

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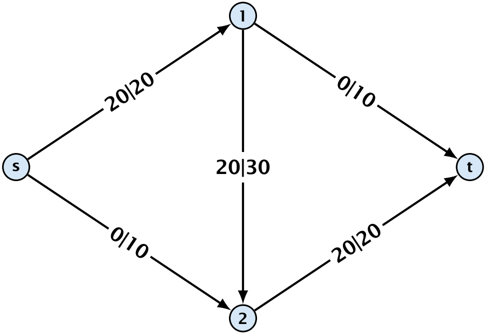

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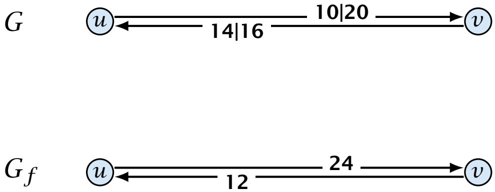

Augmenting Path Algorithm

Definition 1
An augmenting path with respect to flow f, is a path from s to t in the auxiliary graph G_f that contains only edges with non-zero capacity.

Algorithm 44 FordFulkerson($G = (V, E, c)$)

1: Initialize $f(e) \leftarrow 0$ for all edges.
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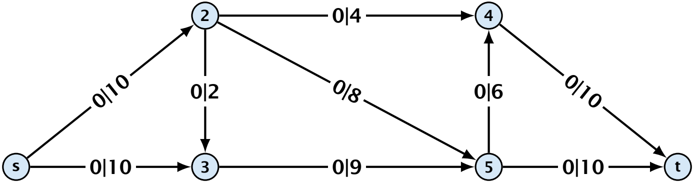
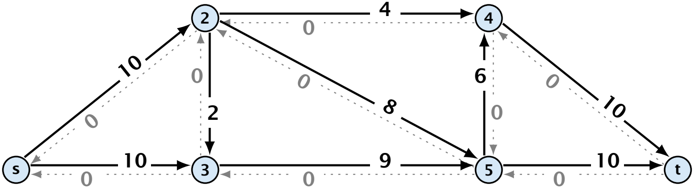


Flow value = 0

G_f



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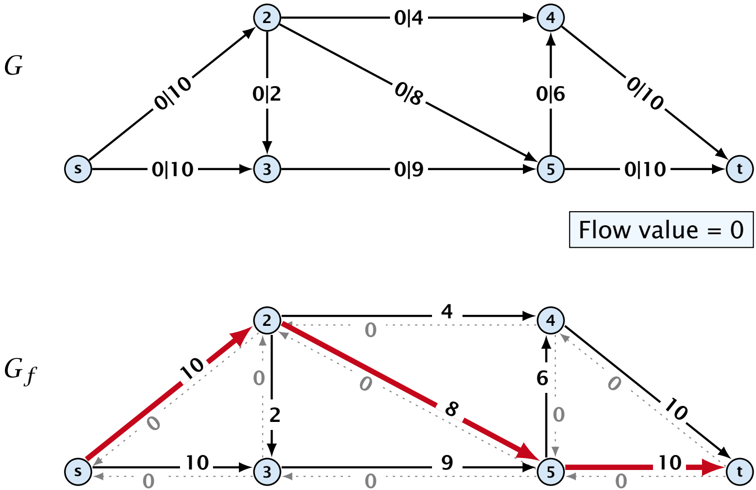

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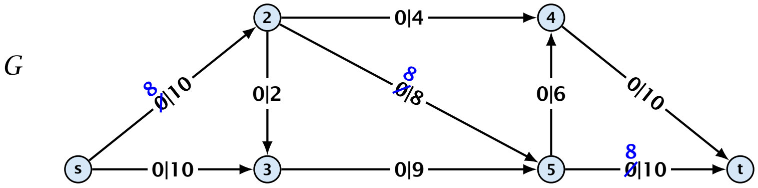
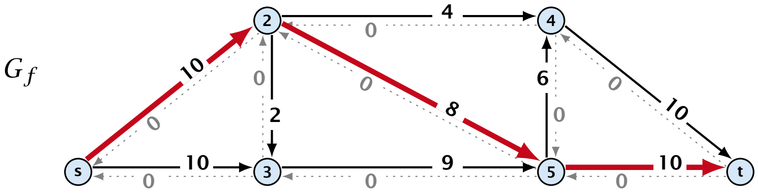


Flow value = 8



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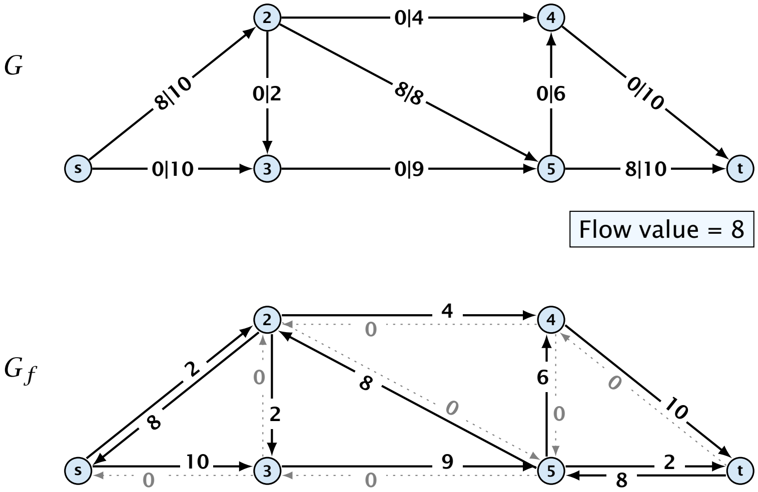

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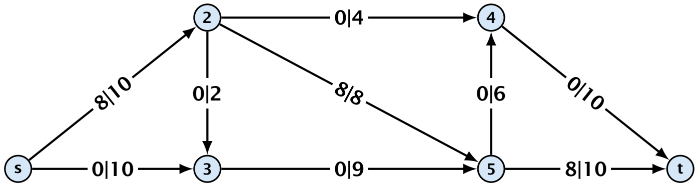

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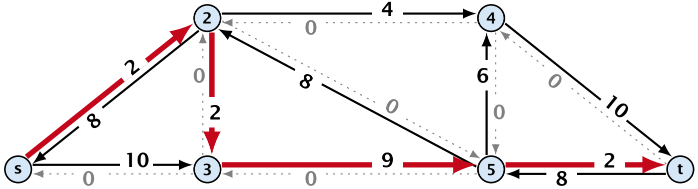

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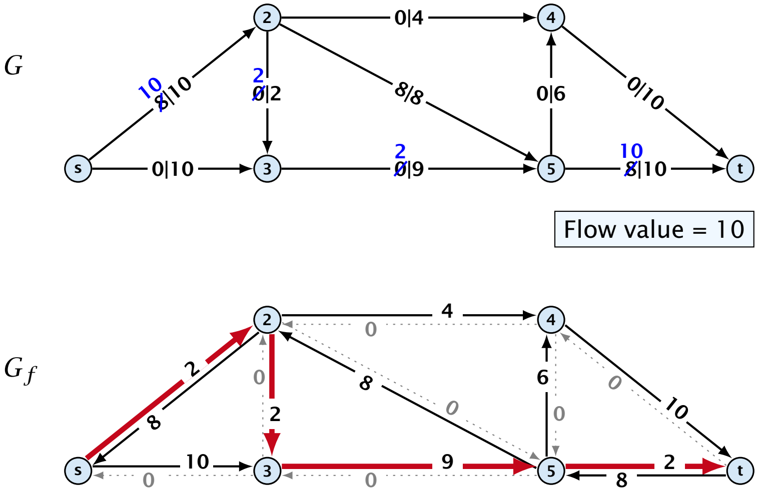

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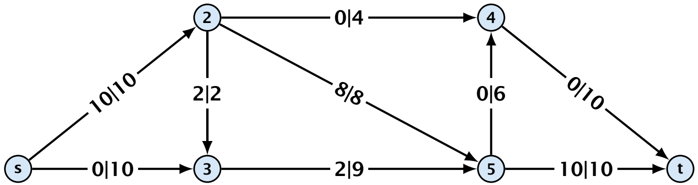
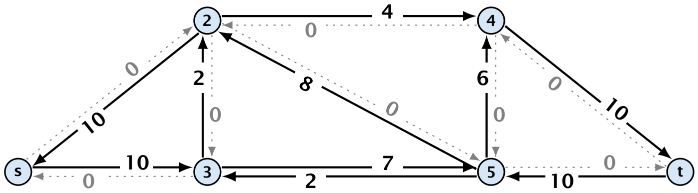


Flow value = 10

G_f



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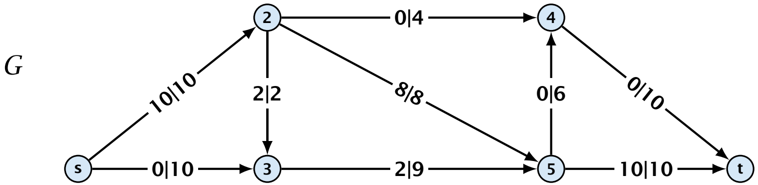

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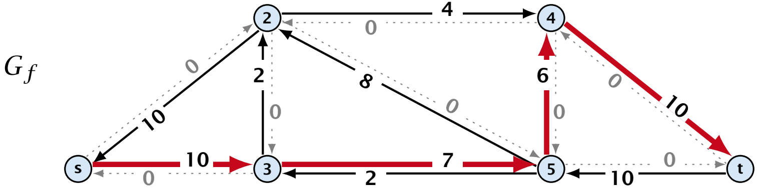

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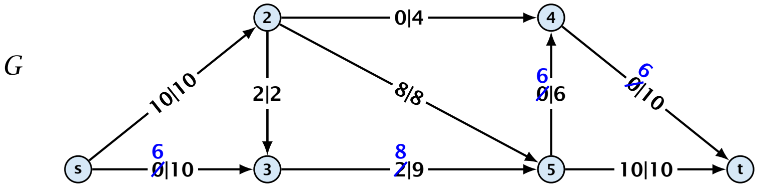
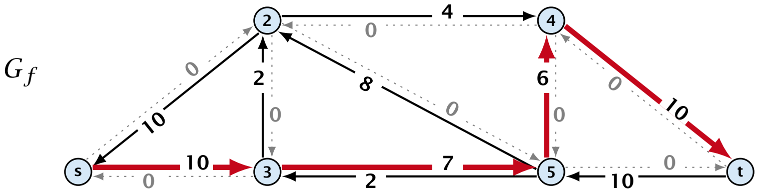


Flow value = 16



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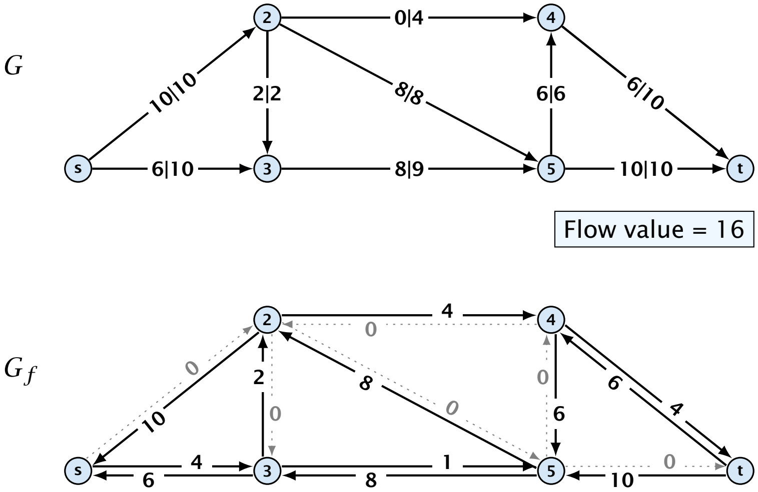

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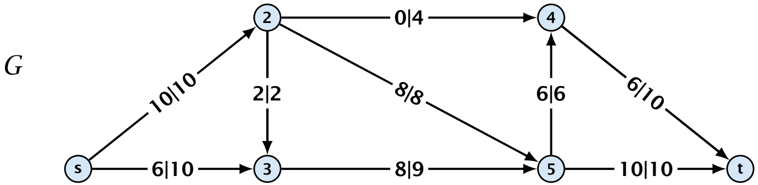

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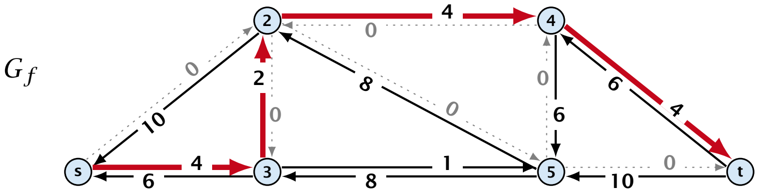

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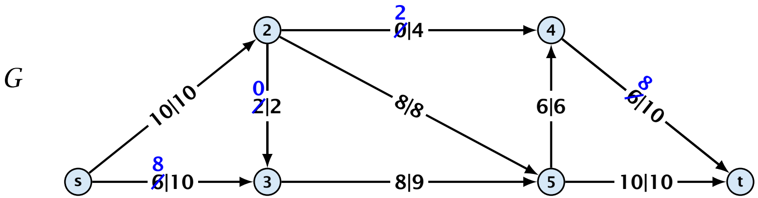
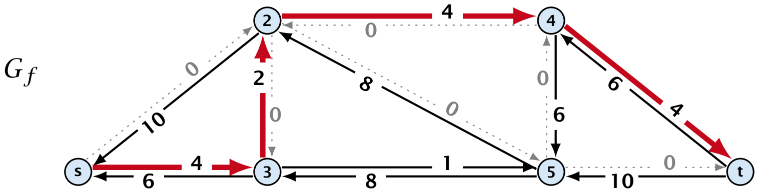


Flow value = 18



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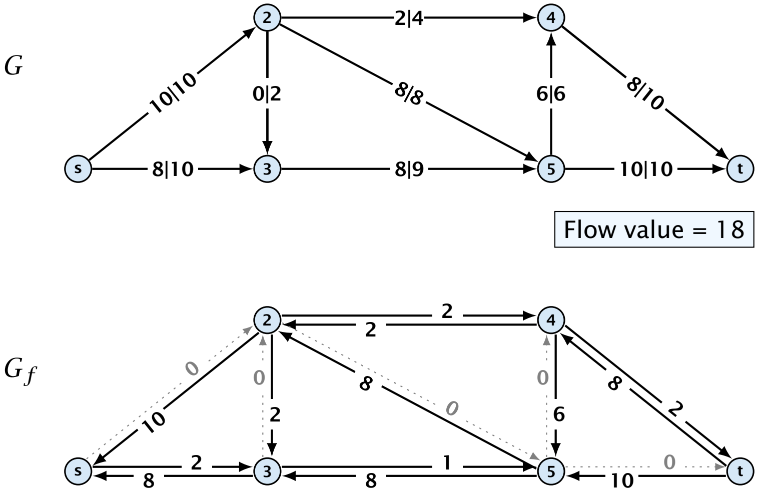

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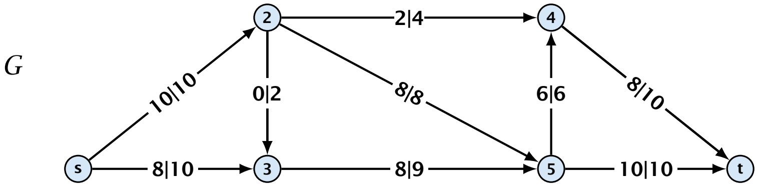

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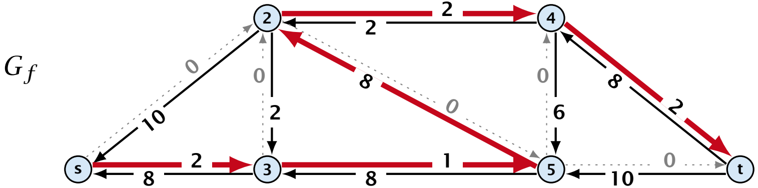

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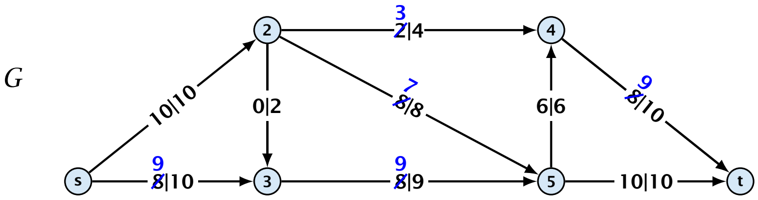

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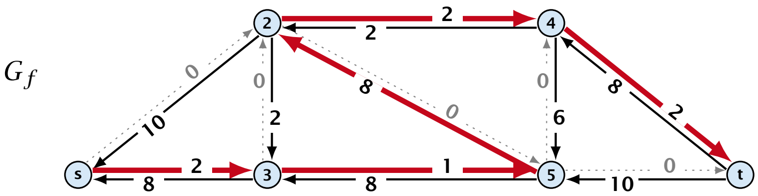

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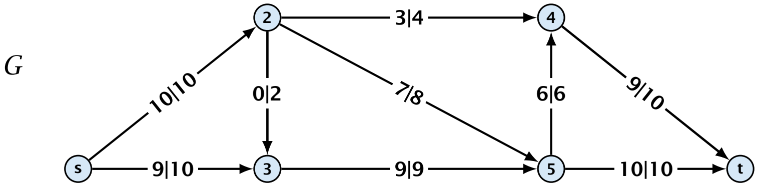

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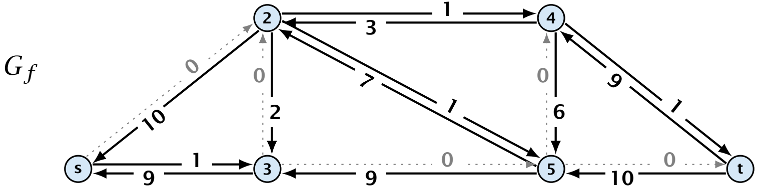

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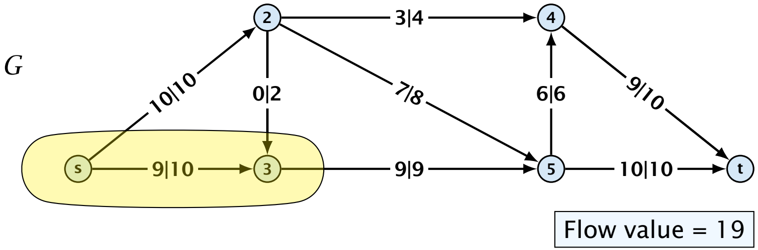

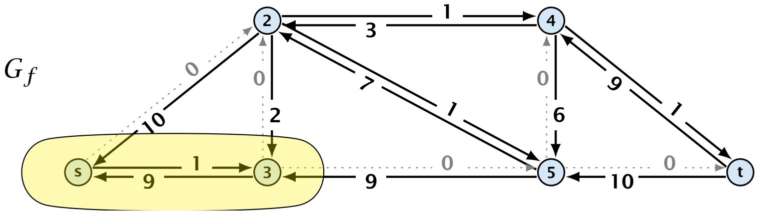

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*A flow f is a maximum flow **iff** there are no augmenting paths.*

Theorem 3

The value of a maximum flow is equal to the value of a minimum cut.

Proof.

Let f be a flow. The following are equivalent:

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This finishes the proof.

Here the first equality uses the flow value lemma, and the second exploits the fact that the flow along incoming edges must be 0 as the residual graph does not have edges leaving A.

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All capacities are integers between 1 and C.

Invariant:
Every flow value $f(e)$ and every residual capacity $c_f(e)$ remains integral troughout the algorithm.

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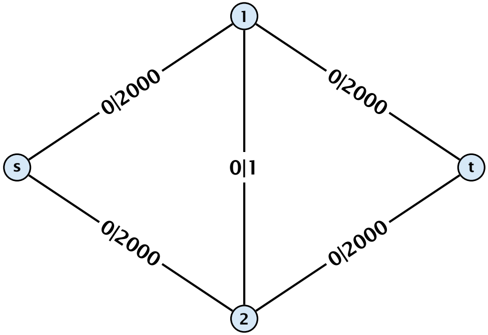
A Bad Input

Problem: The running time may not be polynomial.



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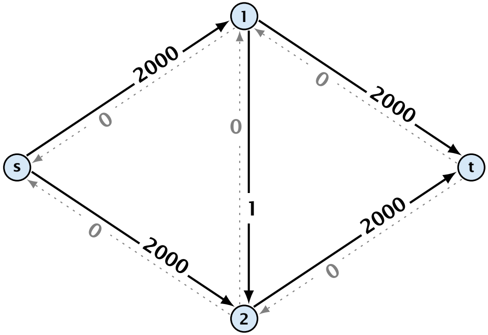

Question:

Can we tweak the algorithm so that the running time is polynomial in the input length?

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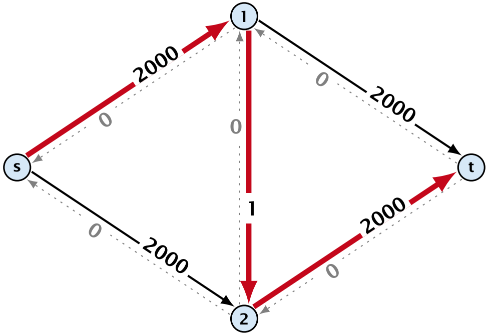

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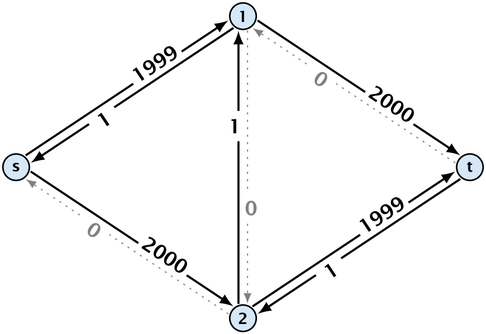

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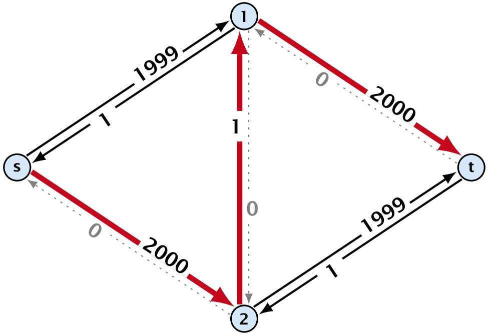

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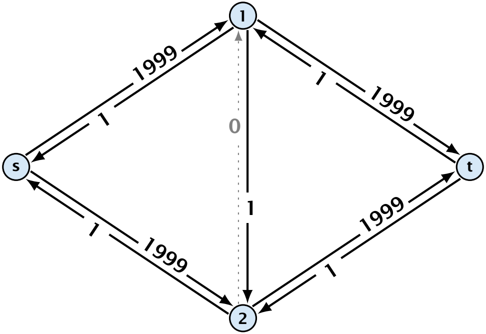

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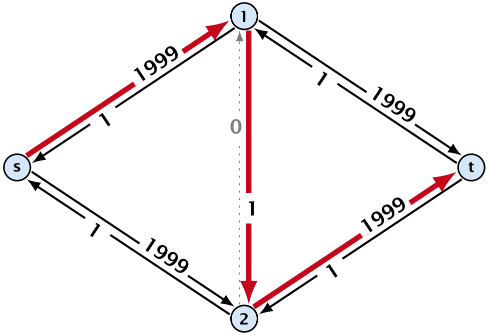

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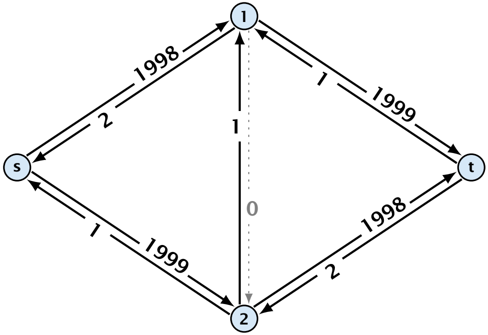

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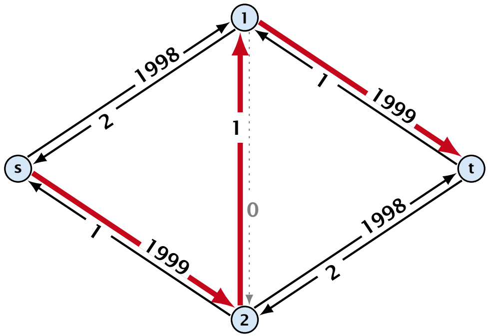

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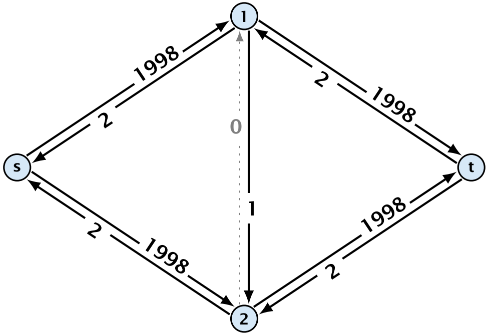

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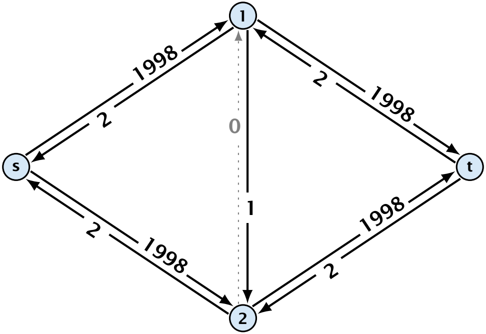

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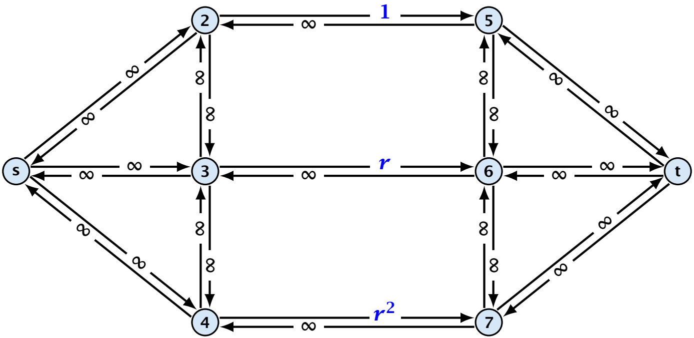

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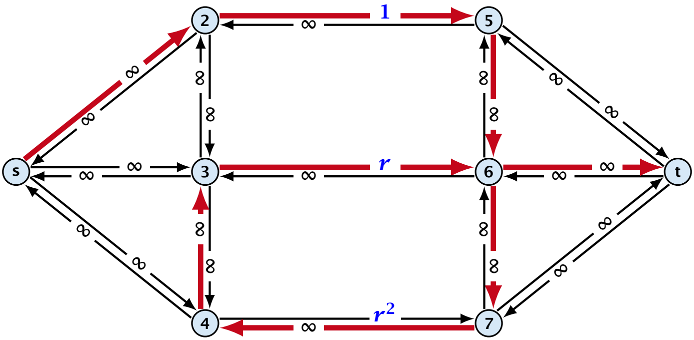

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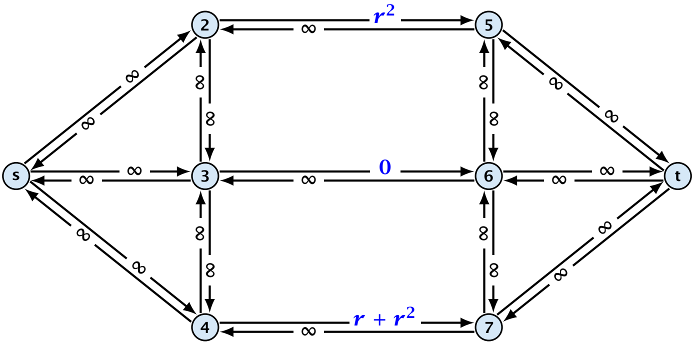

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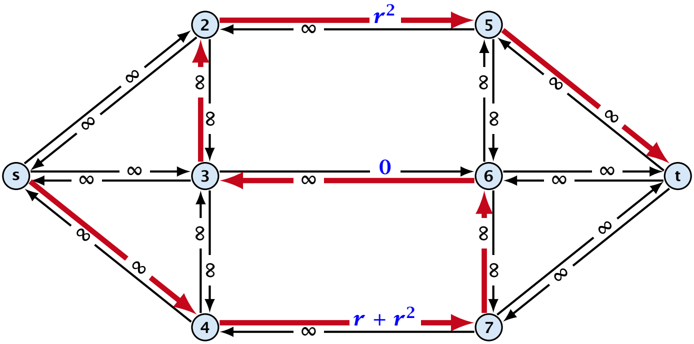

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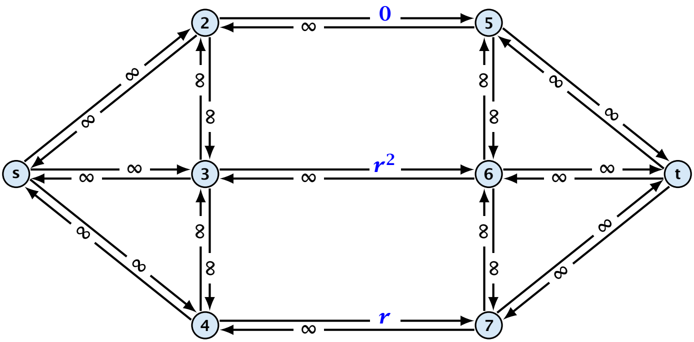

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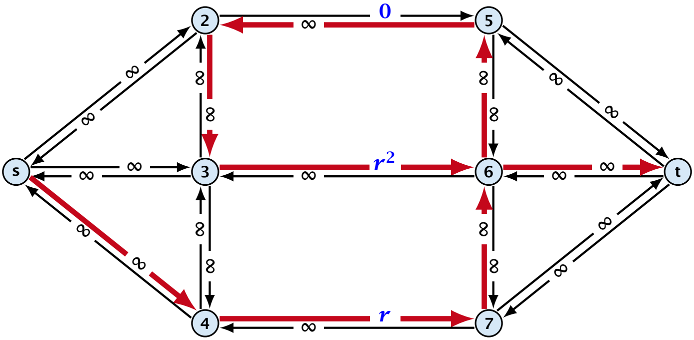

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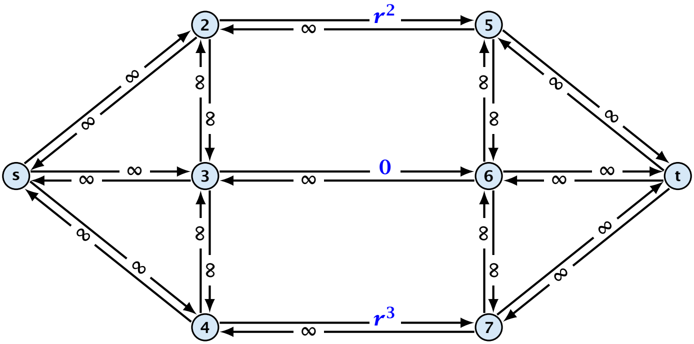

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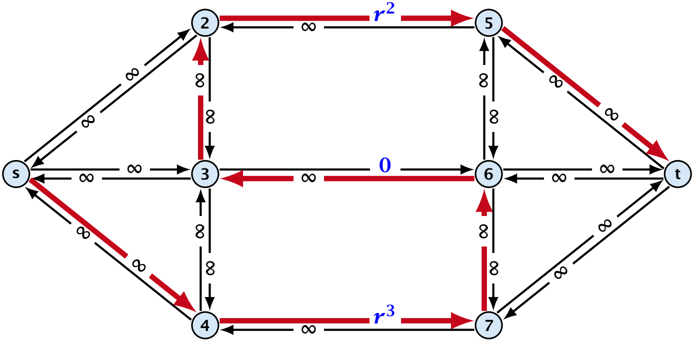

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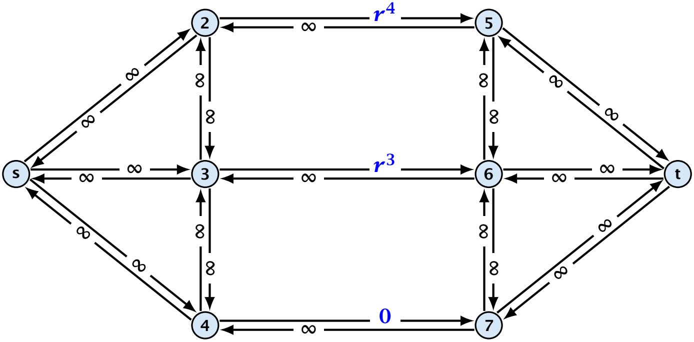

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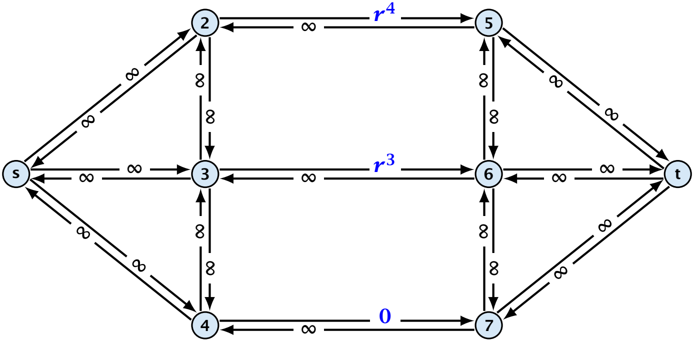

Running time may be infinite!!!

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Overview: Shortest Augmenting Paths

Lemma 6

The length of the shortest augmenting path never decreases.

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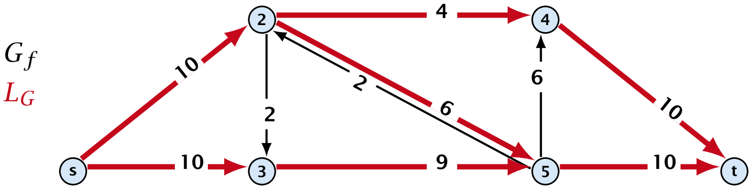

In the following we assume that the residual graph G_f does not contain zero capacity edges.

This means, we construct it in the usual sense and then delete edges of zero capacity.

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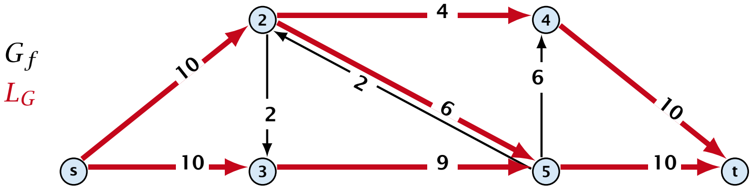

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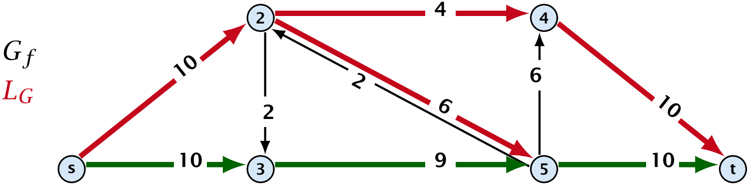

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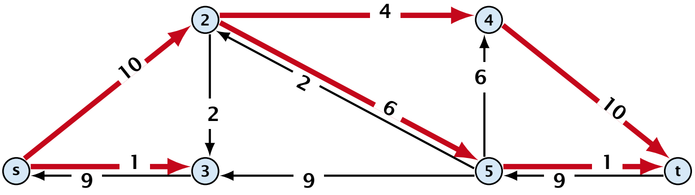

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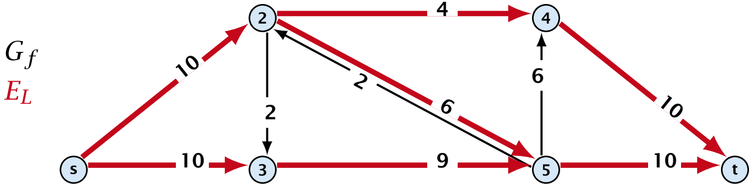

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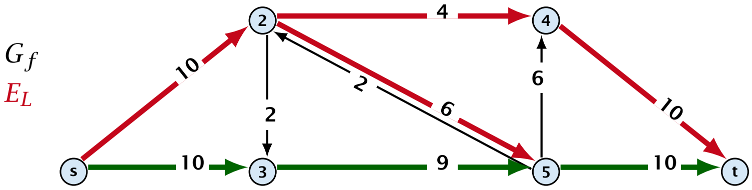

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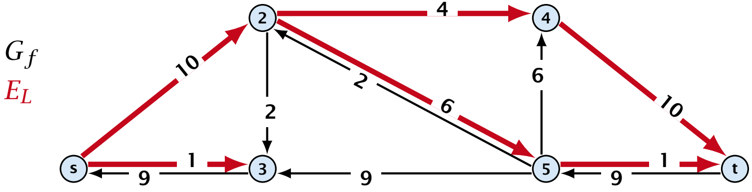

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We maintain a subset E_L of the edges of G_f with the guarantee that a shortest s-t path using only edges from E_L is a shortest augmenting path.

With each augmentation some edges are deleted from E_L.

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Let a phase of the algorithm be defined by the time between two augmentations during which the distance between s and t strictly increases.

Initializing E_L for the phase takes time $\mathcal{O}(m)$.

The total cost for searching for augmenting paths during a phase is at most $\mathcal{O}(mn)$, since every search (successful (i.e., reaching t) or unsuccessful) decreases the number of edges in E_L and takes time $\mathcal{O}(n)$.

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There are at most n phases. Hence, total cost is $\mathcal{O}(mn^2)$.

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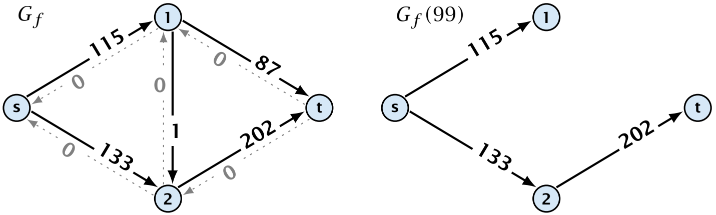

Capacity Scaling

Algorithm 45 maxflow(G, s, t, c)

```
foreach e ∈ E do f_e ← 0;
Δ ← 2^⌈log₂ C⌉
while Δ ≥ 1 do
    G_f(Δ) ← Δ-residual graph
    while there is augmenting path P in G_f(Δ) do
        f ← augment(f, c, P)
        update(G_f(Δ))
    Δ ← Δ/2
return f
```

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- this means we have a maximum flow.

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There are $\lceil \log C \rceil$ *iterations over* Δ.

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- in G_f this cut can have capacity at most $2m\Delta$.
- This gives me an upper bound on the flow that I can still add.

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Theorem 14

We need $\mathcal{O}(m \log C)$ augmentations. The algorithm can be implemented in time $\mathcal{O}(m^2 \log C)$.