## Analysis

**Lemma 8**

*One phase of the Hopcroft-Karp algorithm can be implemented in time $\mathcal{O}(m)$.*

- Do a breadth first search starting at all free vertices in the left side $L$.

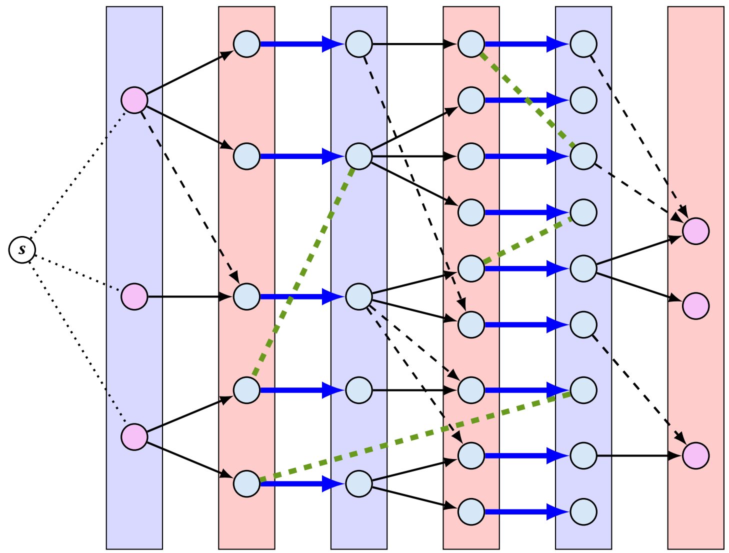
  (alternatively add a super-startnode; connect it to all free vertices in $L$ and start breadth first search from there)
- The search stops when reaching a free vertex. However, the current level of the BFS tree is still finished in order to find a set $F$ of free vertices (on the right side) that can be reached via shortest augmenting paths.



## Analysis

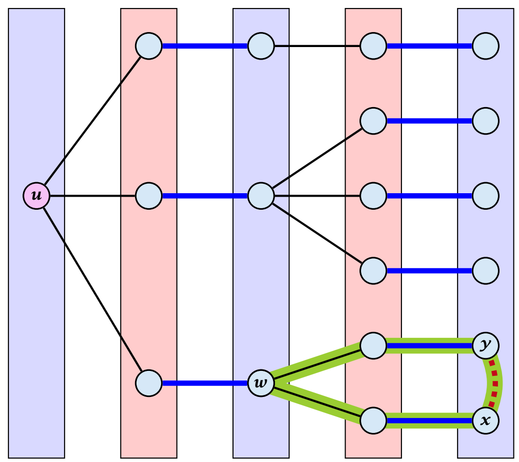
- Then a maxima**l** set of shortest path from the leftmost layer of the tree construction to nodes in $F$ needs to be computed.
- Any such path must visit the layers of the BFS-tree from left to right.
- To go from an odd layer to an even layer it must use a matching edge.
- To go from an even layer to an odd layer edge it can use edges in the BFS-tree **or** edges that have been ignored during BFS-tree construction.
- We direct all edges btw. an even node in some layer $\ell$ to an odd node in layer $\ell + 1$ from left to right.
- A DFS search in the resulting graph gives us a maximal set of vertex disjoint path from left to right in the resulting graph.






## How to find an augmenting path?

**Construct an alternating tree.**




even nodes

odd nodes

**Case 4:**
$y$ is already contained in $T$ as an even vertex

**can't ignore $y$**

The cycle $w \leftrightarrow y - x \leftrightarrow w$ is called a blossom.
$w$ is called the base of the blossom (even node!!!).
The path $u$-$w$ path is called the stem of the blossom.

# Flowers and Blossoms

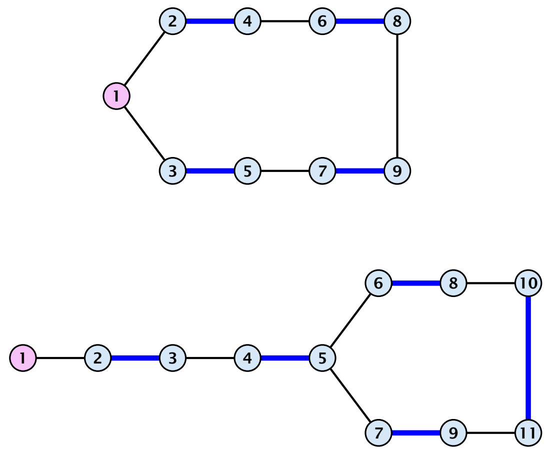
**Definition 9**

A flower in a graph $G = (V, E)$ w.r.t. a matching $M$ and a (free) root node $r$, is a subgraph with two components:

- A stem is an even length alternating path that starts at the root node $r$ and terminates at some node $w$. We permit the possibility that $r = w$ (empty stem).
- A blossom is an odd length alternating cycle that starts and terminates at the terminal node $w$ of a stem and has no other node in common with the stem. $w$ is called the base of the blossom.



# Flowers and Blossoms






# Flowers and Blossoms

**Properties:**

1. A stem spans $2\ell + 1$ nodes and contains $\ell$ matched edges for some integer $\ell \geq 0$.
2. A blossom spans $2k + 1$ nodes and contains $k$ matched edges for some integer $k \geq 1$. The matched edges match all nodes of the blossom except the base.
3. The base of a blossom is an even node (if the stem is part of an alternating tree starting at $r$).

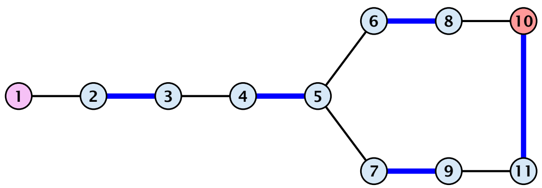


# Flowers and Blossoms

**Properties:**

4. Every node $x$ in the blossom (except its base) is reachable from the root (or from the base of the blossom) through two distinct alternating paths; one with even and one with odd length.
5. The even alternating path to $x$ terminates with a matched edge and the odd path with an unmatched edge.

## Flowers and Blossoms





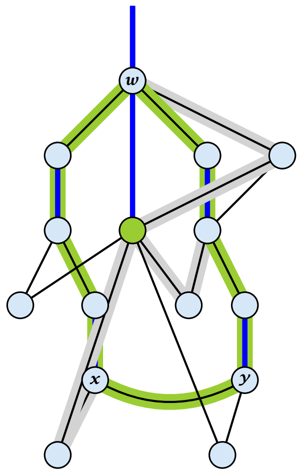

When during the alternating tree construction we discover a blossom $B$ we replace the graph $G$ by $G' = G/B$, which is obtained from $G$ by contracting the blossom $B$.

- Delete all vertices in $B$ (and its incident edges) from $G$.
- Add a new (pseudo-)vertex $b$. The new vertex $b$ is connected to all vertices in $V \setminus B$ that had at least one edge to a vertex from $B$.



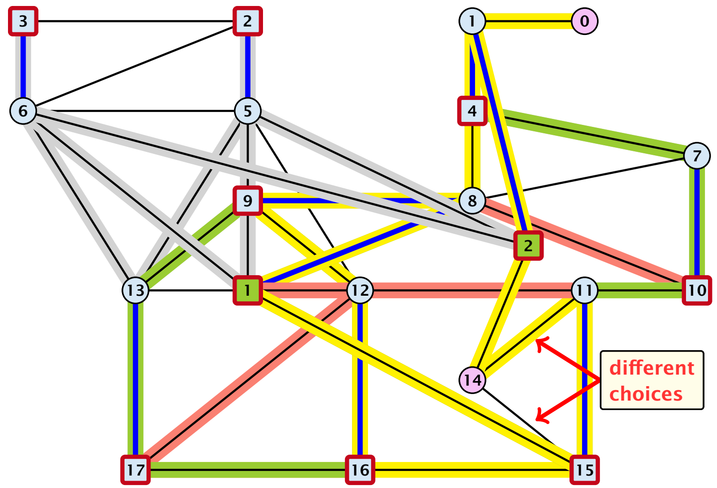
## Shrinking Blossoms

- Edges of $T$ that connect a node $u$ not in $B$ to a node in $B$ become tree edges in $T'$ connecting $u$ to $b$.
- Matching edges (there is at most one) that connect a node $u$ not in $B$ to a node in $B$ become matching edges in $M'$.
- Nodes that are connected in $G$ to at least one node in $B$ become connected to $b$ in $G'$.






## Example: Blossom Algorithm



Assume that we have contracted a blossom $B$ w.r.t. a matching $M$ whose base is $w$. We created graph $G' = G/B$ with pseudonode $b$. Let $M'$ be the matching in the contracted graph.

**Lemma 10**

*If $G'$ contains an augmenting path $p'$ starting at $r$ (or the pseudo-node containing $r$) w.r.t. to the matching $M'$ then $G$ contains an augmenting path starting at $r$ w.r.t. matching $M$.*

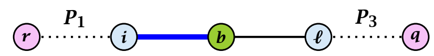


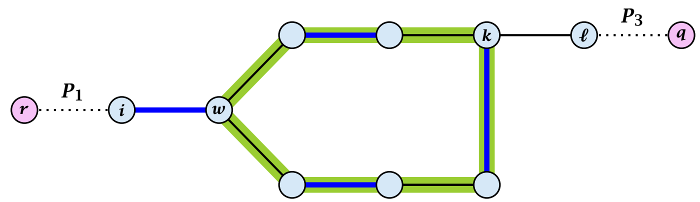
**Proof.**

If $p'$ does not contain $b$ it is also an augmenting path in $G$.

**Case 1: non-empty stem**

- Next suppose that the stem is non-empty.





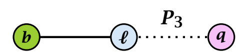

- After the expansion $\ell$ must be incident to some node in the blossom. Let this node be $k$.
- If $k \neq w$ there is an alternating path $P_2$ from $w$ to $k$ that ends in a matching edge.
- $P_1 \circ (i, w) \circ P_2 \circ (k, \ell) \circ P_3$ is an alternating path.
- If $k = w$ then $P_1 \circ (i, w) \circ (w, \ell) \circ P_3$ is an alternating path.

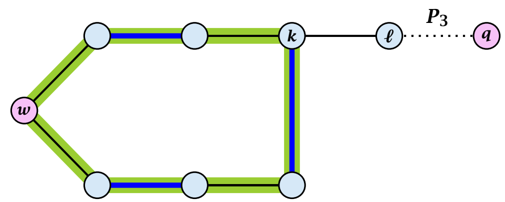


**Proof.**

**Case 2: empty stem**

- If the stem is empty then after expanding the blossom, $w = r$.




- The path $r \circ P_2 \circ (k, \ell) \circ P_3$ is an alternating path.

**Lemma 11**
*If $G$ contains an augmenting path $P$ from $r$ to $q$ w.r.t. matching $M$ then $G'$ contains an augmenting path from $r$ (or the pseudo-node containing $r$) to $q$ w.r.t. $M'$.*



**Proof.**

- If $P$ does not contain a node from $B$ there is nothing to prove.
- We can assume that $r$ and $q$ are the only free nodes in $G$.

**Case 1: empty stem**

Let $i$ be the last node on the path $P$ that is part of the blossom.

$P$ is of the form $P_1 \circ (i,j) \circ P_2$, for some node $j$ and $(i,j)$ is unmatched.

$(b,j) \circ P_2$ is an augmenting path in the contracted network.



**Algorithm 50** search$(r, \mathit{found})$

set $\bar{A}(i) \leftarrow A(i)$ for all nodes $i$
$\mathit{found} \leftarrow \text{false}$
unlabel all nodes;
give an even label to $r$ and initialize $\mathit{list} \leftarrow \{r\}$
**while** $\mathit{list} \neq \emptyset$ **do**
    delete a node $i$ from $\mathit{list}$
    examine$(i, \mathit{found})$
    **if** $\mathit{found} = \text{true}$ **then return**

Search for an augmenting path starting at $r$.
$A(i)$ contains neighbours of node $i$.

We create a copy $\bar{A}(i)$ so that we later can shrink blossoms.

$\mathit{found}$ is just a Boolean that allows to abort the search process

**Algorithm 50** examine$(i, \mathit{found})$

**for** all $j \in \bar{A}(i)$ **do**
    **if** $j$ is even **then** contract$(i,j)$ and **return**
    **if** $j$ is unmatched **then**
        $q \leftarrow j$;
        $\text{pred}(q) \leftarrow i$;
        $\mathit{found} \leftarrow \text{true}$;
        **return**
    **if** $j$ is matched and unlabeled **then**
        $\text{pred}(j) \leftarrow i$;
        $\text{pred}(\text{mate}(j)) \leftarrow j$;
        add $\text{mate}(j)$ to $\mathit{list}$

Examine the neighbours of a node $i$
For all neighbours $j$ do...

You have found a blossom...

You have found a free node which

**Algorithm 50** contract$(i, j)$

trace pred-indices of $i$ and $j$ to identify a blossom $B$
create new node $b$ and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
label $b$ even and add to *list*
update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
form a circular double linked list of nodes in $B$
delete nodes in $B$ from the graph

Contract blossom identified by nodes $i$ and $j$

Get all nodes of the blossom.

Time: $\mathcal{O}(m)$

Identify all neighbours of $b$.

Time: $\mathcal{O}(m)$ (how?)

$b$ will be an even node, and it has

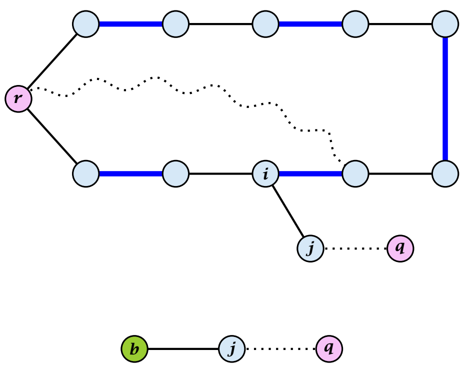


## Analysis

- A contraction operation can be performed in time $\mathcal{O}(m)$. Note, that any graph created will have at most $m$ edges.
- The time between two contraction-operation is basically a BFS/DFS on a graph. Hence takes time $\mathcal{O}(m)$.
- There are at most $n$ contractions as each contraction reduces the number of vertices.
- The expansion can trivially be done in the same time as needed for all contractions.
- An augmentation requires time $\mathcal{O}(n)$. There are at most $n$ of them.
- In total the running time is at most

$$n \cdot (\mathcal{O}(mn) + \mathcal{O}(n)) = \mathcal{O}(mn^2) \ .$$








**Case 2: non-empty stem**

Let $P_3$ be alternating path from $r$ to $w$. Define $M_+ = M \oplus P_3$.

In $M_+$, $r$ is matched and $w$ is unmatched.

$G$ must contain an augmenting path w.r.t. matching $M_+$, since $M$ and $M_+$ have same cardinality.

This path must go between $w$ and $q$ as these are the only unmatched vertices w.r.t. $M_+$.

For $M'_+$ the blossom has an empty stem. Case 1 applies.

$G'$ has an augmenting path w.r.t. $M'_+$. It must also have an augmenting path w.r.t. $M'$, as both matchings have the same cardinality.

This path must go between $r$ and $q$.

## Example: Blossom Algorithm