Part II

Foundations

Vocabularies

$a \cdot b$ "a times b"
"a multiplied by b"
"a into b"

$\frac{a}{b}$ "a divided by b"
"a by b"
"a over b"

(a: numerator (Zähler), b: denominator (Nenner))

a^b "a raised to the b-th power"
"a to the b-th"
"a raised to the power of b"
"a to the power of b"
"a raised to b"
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Vocabularies

$n!$ “n factorial”

$\binom{n}{k}$ “n choose k”

x_i “x subscript i”
“x sub i”
“x i”

$\log_b a$ “log to the base b of a”
“log a to the base b”

$$f : X \to Y, x \mapsto x^2$$

f is a function that maps from domain (Definitionsbereich) X to codomain (Zielmenge) Y. The set $\{y \in Y \mid \exists x \in X : f(x) = y\}$ is the image or the range of the function (Bildbereich/Wertebereich).

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3 Goals

- Gain knowledge about efficient algorithms for important problems, i.e., learn how to solve certain types of problems efficiently.
- Learn how to analyze and judge the efficiency of algorithms.
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What do you measure?

- Memory requirement
- Running time
- Number of comparisons
- Number of multiplications
- Number of hard-disc accesses
- Program size
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How do you measure?

- Implementing and testing on representative inputs
  - How do you choose your inputs?
  - May be very time-consuming.
  - Very reliable results if done correctly.
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- Theoretical analysis in a specific model of computation.

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Input length

The theoretical bounds are usually given by a function $f : \mathbb{N} \to \mathbb{N}$ that maps the input length to the running time (or storage space, comparisons, multiplications, program size etc.).

The input length may e.g. be

- the size of the input (number of bits)
- the number of arguments

Suppose n numbers from the interval $\{1, \ldots, N\}$ have to be sorted. In this case we usually say that the input length is n instead of e.g. $n \log N$, which would be the number of bits required to encode the input.

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Model of Computation

How to measure performance

- Calculate running time and storage space etc. on a simplified, idealized model of computation, e.g. Random Access Machine (RAM), Turing Machine (TM), ...
- Calculate number of certain basic operations: comparisons, multiplications, harddisc accesses, ...

Version is often easier, but focusing on one type of operation makes it more difficult to obtain meaningful results.

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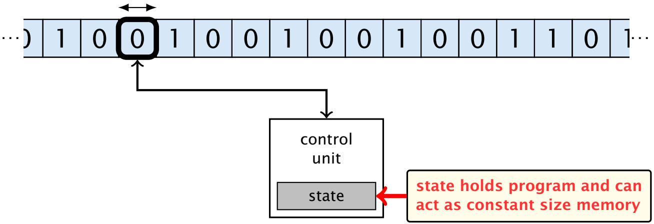
Turing Machine

- Very simple model of computation.
- Only the “current” memory location can be altered.
- Very good model for discussing computabiliy, or polynomial vs. exponential time.
- Some simple problems like recognizing whether input is of the form xx, where x is a string, have quadratic lower bound.

$\Longrightarrow$ **Not a good model for developing efficient algorithms.**



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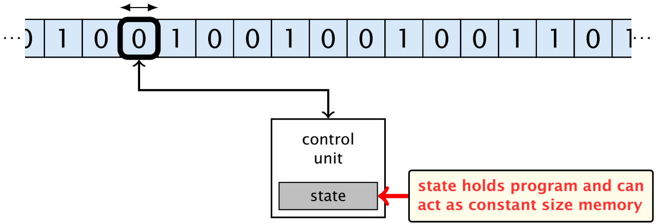

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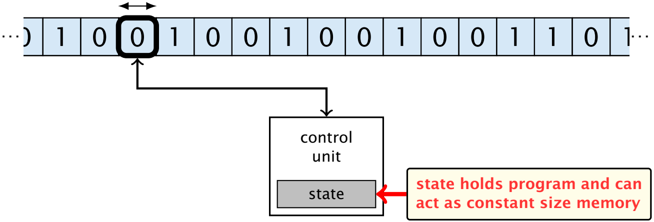

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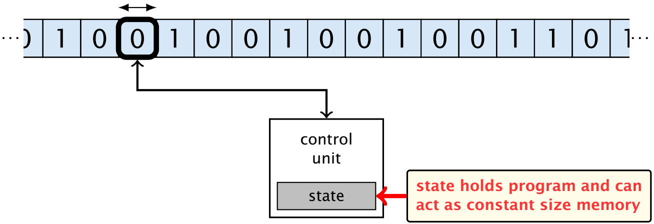

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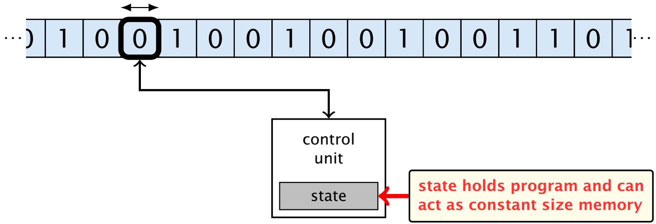
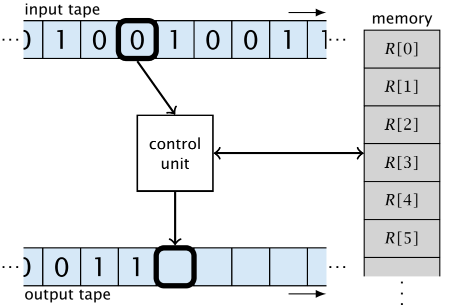


Random Access Machine (RAM)

- Input tape and output tape (sequences of zeros and ones; unbounded length).
- Memory unit: infinite but countable number of registers $R[0], R[1], R[2], \ldots$.
- Registers hold integers.
- Indirect addressing.



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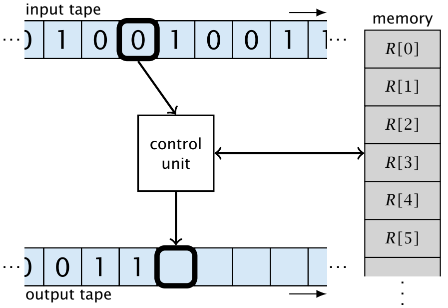

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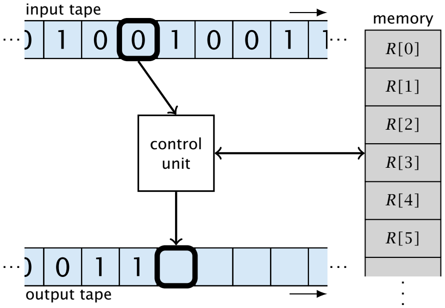

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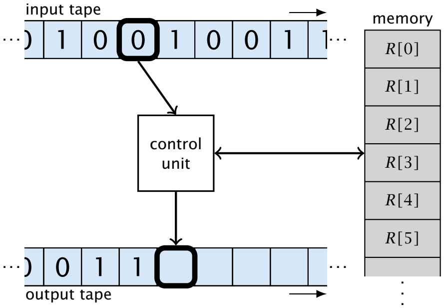

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Operations

- input operations (input tape → $R[i]$)
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- register-register transfers
  - $R[j] := R[i]$
  - $R[j] := 4$
- **indirect** addressing
  - $R[j] := R[R[i]]$
    loads the content of the $R[i]$-th register into the j-th register
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- branching (including loops) based on comparisons
  - `jump` x
    jumps to position x in the program;
    sets instruction counter to x;
    reads the next operation to perform from register $R[x]$
  - `jumpz` x $R[i]$
    jump to x if $R[i] = 0$
    if not the instruction counter is increased by 1;
  - `jumpi` i
    jump to $R[i]$ (indirect jump);
- arithmetic instructions: $+$, $-$, $\times$, $/$
  - $R[i] := R[j] + R[k];$
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Random Access Machine (RAM)

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- branching (including loops) based on comparisons
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- uniform cost model
  Every operation takes time 1.

- logarithmic cost model
  The cost depends on the content of memory cells:
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4 Modelling Issues

Example 2

Algorithm 1 RepeatedSquaring(n)

$r \leftarrow 2$;
for $i = 1 \to n$ **do**
    $r \leftarrow r^2$
return r

running time

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There are different types of complexity bounds:

- best-case complexity:

$$C_{\mathrm{bc}}(n) := \min\{C(x) \mid |x| = n\}$$

  Usually easy to analyze, but not very meaningful.

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Formal Definition

Let f denote functions from $\mathbb{N}$ to $\mathbb{R}^+$.

- $\mathcal{O}(f) = \{g \mid \exists c > 0\ \exists n_0 \in \mathbb{N}_0\ \forall n \geq n_0 : [g(n) \leq c \cdot f(n)]\}$
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There is an equivalent definition using limes notation (assuming that the respective limes exists). f and g are functions from $\mathbb{N}_0$ to $\mathbb{R}_0^+$.

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Asymptotic Notation in Equations

How do we interpret an expression like:

$$2n^2 + 3n + 1 = 2n^2 + \Theta(n)$$

Here, $\Theta(n)$ stands for an anonymous function in the set $\Theta(n)$ that makes the expression true.

Note that $\Theta(n)$ is on the right hand side, otw. this interpretation is wrong.

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"It is understood" that every occurence of an $\mathcal{O}$-symbol (or $\Theta, \Omega, o, \omega$) on the left represents one anonymous function.

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We can view an expression containing asymptotic notation as generating a set:

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n)$$

represents

$$\Big\{f : \mathbb{N} \to \mathbb{R}^+ \mid f(n) = n^2 \cdot g(n) + h(n)$$
$$\text{with } g(n) \in \mathcal{O}(n) \text{ and } h(n) \in \mathcal{O}(\log n)\Big\}$$

Asymptotic Notation in Equations

Then an asymptotic equation can be interpreted as containement btw. two sets:

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n) = \Theta(n^2)$$

represents

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n) \subseteq \Theta(n^2)$$

Asymptotic Notation

Lemma 3

Let f, g be functions with the property
$\exists n_0 > 0\, \forall n \geq n_0 : f(n) > 0$ *(the same for g). Then*

- $c \cdot f(n) \in \Theta(f(n))$ *for any constant c*
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Let f, g be functions with the property
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In general asymptotic classification of running times is a good measure for comparing algorithms:

- If the running time analysis is tight and actually occurs in practise (i.e., the asymptotic bound is not a purely theoretical worst-case bound), then the algorithm that has better asymptotic running time will always outperform a weaker algorithm for large enough values of n.
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$n \leftarrow \text{size}(L)$
if $n \le 1$ **return** L
$L_1 \leftarrow L[1 \cdots \lfloor \frac{n}{2} \rfloor]$
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This algorithm requires

$$T(n) = T\left(\left\lceil \frac{n}{2} \right\rceil\right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \mathcal{O}(n) \le 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + \mathcal{O}(n)$$

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Methods for Solving Recurrences

1. **Guessing+Induction**
   Guess the right solution and prove that it is correct via induction. It needs experience to make the right guess.
2. **Master Theorem**
   For a lot of recurrences that appear in the analysis of algorithms this theorem can be used to obtain tight asymptotic bounds. It does not provide exact solutions.
3. **Characteristic Polynomial**
   Linear homogenous recurrences can be solved via this method.

Methods for Solving Recurrences

4. **Generating Functions**
   A more general technique that allows to solve certain types of linear inhomogenous relations and also sometimes non-linear recurrence relations.
5. **Transformation of the Recurrence**
   Sometimes one can transform the given recurrence relations so that it e.g. becomes linear and can therefore be solved with one of the other techniques.

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First we need to get rid of the $\mathcal{O}$-notation in our recurrence:

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Formally one would make an induction proof, where the above is the induction step. The base case is usually trivial.

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  Hence, statement is true if we choose $d \geq c$.

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Note that we can do this as for constant-sized inputs the running time is always some constant (b in the above case).

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We also make a guess of $T(n) \le dn \log n$ and get

$$
\begin{aligned}
T(n) &\le 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn \\
&\le 2\left(d\left\lceil \frac{n}{2} \right\rceil \log \left\lceil \frac{n}{2} \right\rceil\right) + cn \\
\boxed{\left\lceil \tfrac{n}{2} \right\rceil \le \tfrac{n}{2} + 1} \quad &\le 2\big(d(n/2+1)\log(n/2+1)\big) + cn \\
\boxed{\tfrac{n}{2} + 1 \le \tfrac{9}{16}n} \quad &\le dn \log\left(\frac{9}{16}n\right) + 2d\log n + cn \\
\boxed{\log \tfrac{9}{16}n = \log n + (\log 9 - 4)} \quad &= dn \log n + (\log 9 - 4)dn + 2d\log n + cn \\
\boxed{\log n \le \tfrac{n}{4}} \quad &
\end{aligned}
$$

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\boxed{\left\lceil \tfrac{n}{2} \right\rceil \le \tfrac{n}{2} + 1} \quad &\le 2\big(d(n/2+1)\log(n/2+1)\big) + cn \\
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\boxed{\log n \le \tfrac{n}{4}} \quad &\le dn\log n + (\log 9 - 3.5)dn + cn
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$$

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\boxed{\left\lceil \tfrac{n}{2}\right\rceil \leq \tfrac{n}{2}+1} \quad &\leq 2(d(n/2+1)\log(n/2+1)) + cn \\
\boxed{\tfrac{n}{2}+1 \leq \tfrac{9}{16}n} \quad &\leq dn\log\left(\frac{9}{16}n\right) + 2d\log n + cn \\
\boxed{\log\tfrac{9}{16}n = \log n + (\log 9 - 4)} \quad &= dn\log n + (\log 9 - 4)dn + 2d\log n + cn \\
\boxed{\log n \leq \tfrac{n}{4}} \quad &\leq dn\log n + (\log 9 - 3.5)dn + cn \\
&\leq dn\log n - 0.33dn + cn
\end{aligned}
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\boxed{\log n \leq \tfrac{n}{4}} \quad &\leq dn \log n + (\log 9 - 3.5)dn + cn \\
&\leq dn \log n - 0.33dn + cn \\
&\leq dn \log n
\end{aligned}
$$

for a suitable choice of d.

6.2 Master Theorem

Lemma 4

Let $a \geq 1, b \geq 1$ *and* $\epsilon > 0$ *denote constants. Consider the recurrence*

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) \ .$$

Case 1.

If $f(n) = \mathcal{O}(n^{\log_b(a)-\epsilon})$ *then* $T(n) = \Theta(n^{\log_b a})$.

Case 2.

If $f(n) = \Theta(n^{\log_b(a)} \log^k n)$ *then* $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$, $k \geq 0$.

Case 3.

If $f(n) = \Omega(n^{\log_b(a)+\epsilon})$ *and for sufficiently large* n $af(\frac{n}{b}) \leq cf(n)$ *for some constant* $c < 1$ *then* $T(n) = \Theta(f(n))$.

6.2 Master Theorem

We prove the Master Theorem for the case that n is of the form b^ℓ, and we assume that the non-recursive case occurs for problem size 1 and incurs cost 1.

The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

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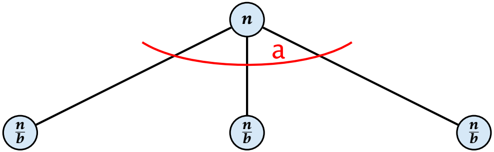

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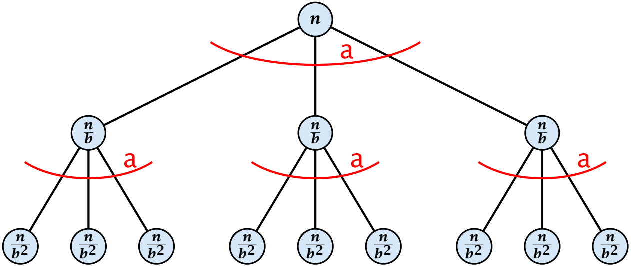

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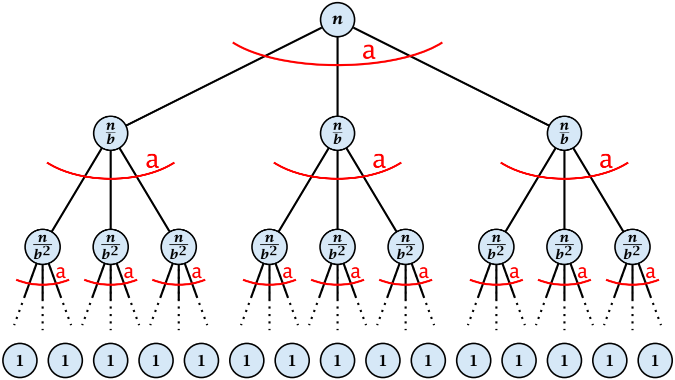

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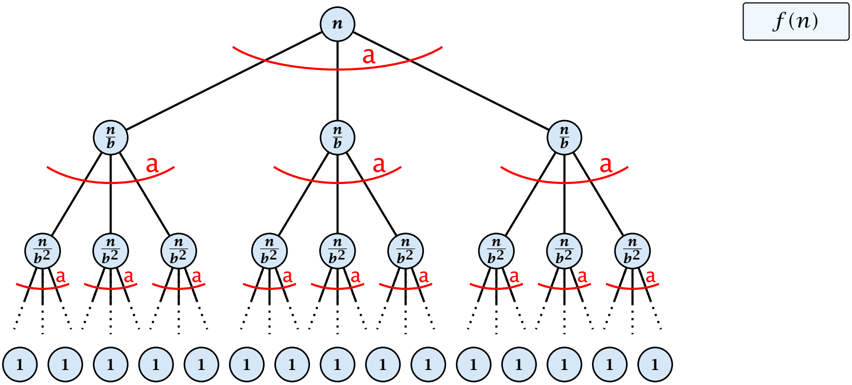

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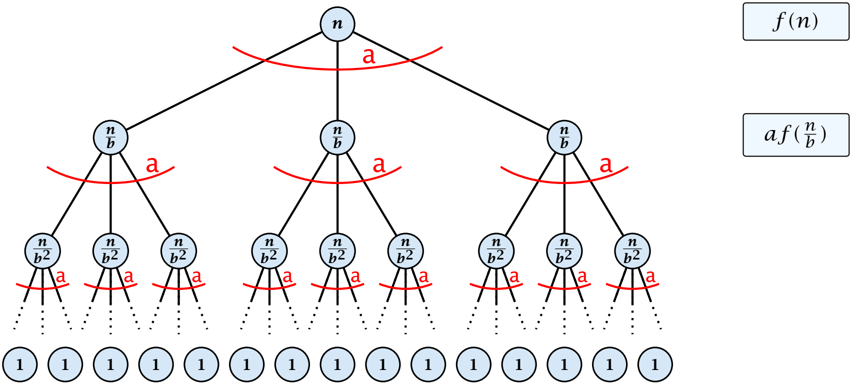

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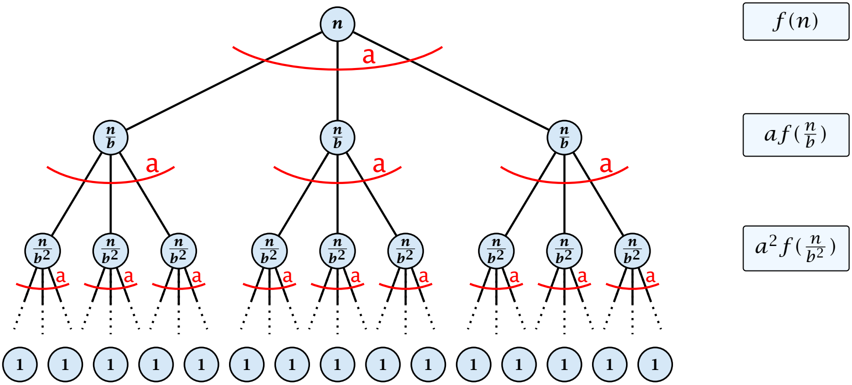

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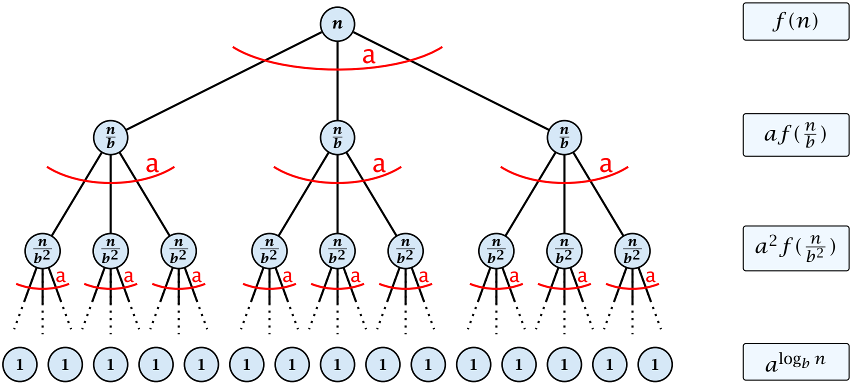

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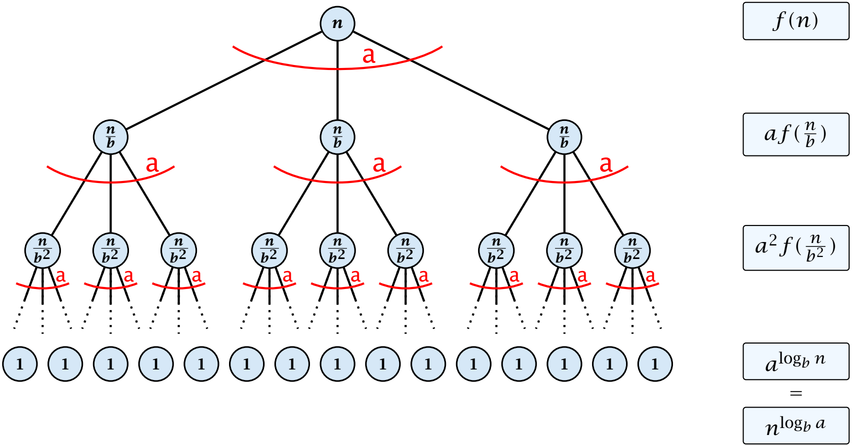

6.2 Master Theorem

This gives

$$T(n) = n^{\log_b a} + \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \ .$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

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$$T(n) - n^{\log_b a}$$

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Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon}
\end{aligned}
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$$\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}}$$

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&\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a-\epsilon} \\
&= cn^{\log_b a-\epsilon} \sum_{i=0}^{\log_b n-1} \left(b^{\epsilon}\right)^i
\end{aligned}
$$

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$$\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}} \quad = cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^{\epsilon})^i$$

$$\boxed{\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}}$$

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T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \\
\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}} \quad &= cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^{\epsilon})^i \\
\boxed{\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}} \quad &= cn^{\log_b a - \epsilon}(b^{\epsilon \log_b n} - 1)/(b^{\epsilon} - 1)
\end{aligned}
$$

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \\ \boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}} \quad &= cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} \left(b^{\epsilon}\right)^i \\ \boxed{\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}} \quad &= cn^{\log_b a - \epsilon}(b^{\epsilon \log_b n} - 1)/(b^{\epsilon} - 1) \\ &= cn^{\log_b a - \epsilon}(n^{\epsilon} - 1)/(b^{\epsilon} - 1) \end{aligned}$$

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&= cn^{\log_b a - \epsilon}(n^{\epsilon} - 1)/(b^{\epsilon} - 1) \\
&= \frac{c}{b^{\epsilon} - 1} n^{\log_b a}(n^{\epsilon} - 1)/(n^{\epsilon})
\end{aligned}$$

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&= cn^{\log_b a-\epsilon}(n^{\epsilon} - 1)/(b^{\epsilon} - 1) \\
&= \frac{c}{b^{\epsilon} - 1} n^{\log_b a}(n^{\epsilon} - 1)/(n^{\epsilon})
\end{aligned}
$$

Hence,

$$
T(n) \leq \left(\frac{c}{b^{\epsilon} - 1} + 1\right) n^{\log_b(a)}
$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}} \qquad \boxed{\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}}$$

$$\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \\
&= cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} \left(b^{\epsilon}\right)^i \\
&= cn^{\log_b a - \epsilon}(b^{\epsilon \log_b n} - 1)/(b^{\epsilon} - 1) \\
&= cn^{\log_b a - \epsilon}(n^{\epsilon} - 1)/(b^{\epsilon} - 1) \\
&= \frac{c}{b^{\epsilon} - 1} n^{\log_b a}(n^{\epsilon} - 1)/(n^{\epsilon})
\end{aligned}$$

Hence,

$$T(n) \leq \left(\frac{c}{b^{\epsilon} - 1} + 1\right) n^{\log_b (a)}$$

$$\boxed{\Rightarrow T(n) = \mathcal{O}(n^{\log_b a}).}$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}$.

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T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\
&= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\
&= cn^{\log_b a} \log_b n
\end{aligned}
$$

Case 2. Now suppose that $f(n) \leq c n^{\log_b a}$.

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&= c n^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\
&= c n^{\log_b a} \log_b n
\end{aligned}
$$

Hence,

$$
T(n) = \mathcal{O}(n^{\log_b a} \log_b n)
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T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
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&= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\
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\end{aligned}$$

Hence,

$$T(n) = \mathcal{O}(n^{\log_b a} \log_b n) \qquad \boxed{\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log n).}$$

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&= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\
&= cn^{\log_b a} \log_b n
\end{aligned}$$

Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n)$$

Case 2. Now suppose that $f(n) \geq cn^{\log_b a}$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\
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&= cn^{\log_b a} \sum_{i=0}^{\log_b n-1} 1 \\
&= cn^{\log_b a} \log_b n
\end{aligned}
$$

Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n) \qquad \boxed{\Rightarrow T(n) = \Omega(n^{\log_b a} \log n).}$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}(\log_b(n))^k$.

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

Case 2. Now suppose that $f(n) \le c n^{\log_b a} (\log_b(n))^k$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k
\end{aligned}
$$

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\end{aligned}
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$\boxed{n = b^\ell \Rightarrow \ell = \log_b n}$

Case 2. Now suppose that $f(n) \le cn^{\log_b a}(\log_b(n))^k$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k$$

$\boxed{n = b^\ell \Rightarrow \ell = \log_b n}$

$$= cn^{\log_b a} \sum_{i=0}^{\ell - 1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}(\log_b(n))^k$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \\
\boxed{n = b^\ell \Rightarrow \ell = \log_b n} \qquad &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\
&= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k
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&= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \\
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&= c n^{\log_b a} \sum_{i=0}^{\ell - 1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\
&= c n^{\log_b a} \sum_{i=0}^{\ell - 1} (\ell - i)^k \\
&= c n^{\log_b a} \sum_{i=1}^{\ell} i^k \\
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\end{aligned}$$

$\boxed{n = b^\ell \Rightarrow \ell = \log_b n}$

$\boxed{\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log^{k+1} n).}$

Case 3. Now suppose that $f(n) \geq dn^{\log_b a+\epsilon}$, and that for sufficiently large n: $af(n/b) \leq cf(n)$, for $c < 1$.

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)$$

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n-1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

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$$\boxed{q < 1 : \sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}}$$

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\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq \sum_{i=0}^{\log_b n-1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\
&\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a})
\end{aligned}
$$

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&\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a})
\end{aligned}
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Hence,

$$
T(n) \leq \mathcal{O}(f(n))
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\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq \sum_{i=0}^{\log_b n-1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\
&\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a})
\end{aligned}
$$

$\boxed{q < 1 : \sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}}$

Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

$\boxed{\Rightarrow T(n) = \Theta(f(n)).}$

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

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1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & \boxed{1} & A \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & \boxed{1} & B \\
\hline
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1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & \quad A \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1_{1} & 1 & \quad B \\
\hline
 & & & & & & & & 0 &
\end{array}$$

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1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & & B \\
 & & & & & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & \\
\hline
 & & & & & & 0 & 0 & 0 & &
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1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & & B \\
 & & & & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & \\
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 & & & & & 1 & 0 & 0 & 0 & &
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\hline
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 & & & {}_1 & {}_0 & {}_1 & {}_1 & {}_1 & & & \\
\hline
 & & & & 0 & 1 & 0 & 0 & 0 & &
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1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & & B \\
 & & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & \\
\hline
 & & & 0 & 0 & 1 & 0 & 0 & 0 & &
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1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & & B \\
 & & {}_1 & {}_1 & {}_0 & {}_1 & {}_1 & {}_1 & & & \\
\hline
 & & & 0 & 0 & 1 & 0 & 0 & 0 & &
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1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\
 & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & \\
\hline
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1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & & B \\
 & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & \\
\hline
 & & 1 & 0 & 0 & 1 & 0 & 0 & 0 & &
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1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & & B\\
{}_0 & {}_0 & {}_1 & {}_1 & {}_0 & {}_1 & {}_1 & {}_1 & & & \\ \hline
 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & &
\end{array}$$

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1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
0	0	1	1	0	1	1	1		
	1	1	0	0	1	0	0	0	

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 & \boxed{1} & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & & A \\
 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & & B \\
{\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & \\
\hline
 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & &
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$$
\begin{array}{c|ccccccccc c}
 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\
 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\
 {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & \\
\hline
 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 &
\end{array}
$$

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$$
\begin{array}{cccccccccccc}
 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & & A \\
 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & & B \\
{\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & \\
\hline
\boxed{1} & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & &
\end{array}
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 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & & B \\
{\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & \\
\hline
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & &
\end{array}
$$

This gives that two n-bit integers can be added in time $\mathcal{O}(n)$.

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

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$$\begin{array}{c} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline \end{array}$$

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Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

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\begin{array}{ccccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & \boxed{1} \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1
\end{array}
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\begin{array}{ccccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & \boxed{1} & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1
\end{array}
$$

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$$
\begin{array}{ccccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & \boxed{1} & 1 \\
\hline
 & & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & & & & & & & 0
\end{array}
$$

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$$
\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & \boxed{1} & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0
\end{array}
$$

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$$
\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & \boxed{0} & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0
\end{array}
$$

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$$
\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & \boxed{0} & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & & & & & & 0 & 0
\end{array}
$$

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$$
\begin{array}{ccccccccccc}
 & & 1 & 0 & 0 & 0 & 1 & \times & 1 & \boxed{0} & 1 & 1 \\
\hline
 & & & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

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Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

$$
\begin{array}{ccccccccccc}
 & & & & 1 & 0 & 0 & 0 & 1 & \times & \boxed{1} & 0 & 1 & 1 \\
\hline
 & & & & & & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & & & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

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 & & & 1 & 0 & 0 & 0 & 1 \\
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 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

$$
\begin{array}{rrrrrrrrrr}
 & 1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 \\
\hline
 & & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
\end{array}
$$

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Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

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 & & & & & 1 & 0 & 0 & 0 & 1 \\
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 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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\hline
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\end{array}
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Time requirement:

Example: Multiplying Two Integers

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Time requirement:

- Computing intermediate results: $\mathcal{O}(nm)$.

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\hline
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$$

$$
\begin{array}{ccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 \; 1\\
\end{array}
$$

Time requirement:

- Computing intermediate results: $\mathcal{O}(nm)$.
- Adding m numbers of length $\le 2n$:
  $\mathcal{O}((m+n)m) = \mathcal{O}(nm)$.

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k.

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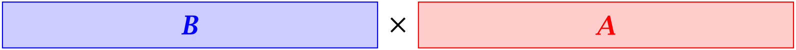

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b_n	$\cdots$	b_0	$\times$	a_n	$\cdots$	a_0

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b_n	$\cdots$	$b_{\frac{n}{2}}$	$b_{\frac{n}{2}-1}$	$\cdots$	b_0	$\times$	a_n	$\cdots$	$a_{\frac{n}{2}}$	$a_{\frac{n}{2}-1}$	$\cdots$	a_0

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Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

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Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

Hence,

$$A \cdot B = A_1 B_1 \cdot 2^n + (A_1 B_0 + A_0 B_1) \cdot 2^{\frac{n}{2}} + A_0 \cdot B_0$$

Example: Multiplying Two Integers

Algorithm 3 mult(A, B)

```
if |A| = |B| = 1 then
    return a_0 · b_0
split A into A_0 and A_1
split B into B_0 and B_1
Z_2 ← mult(A_1, B_1)
Z_1 ← mult(A_1, B_0) + mult(A_0, B_1)
Z_0 ← mult(A_0, B_0)
return Z_2 · 2^n + Z_1 · 2^(n/2) + Z_0
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```

1: **if** $|A| = |B| = 1$ **then**

2: **return** $a_0 \cdot b_0$

3: split A into A_0 and A_1

4: split B into B_0 and B_1

5: $Z_2 \leftarrow \mathrm{mult}(A_1, B_1)$

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$\mathcal{O}(1)$

Example: Multiplying Two Integers

Algorithm 3 mult(A, B)

if $|A| = |B| = 1$ **then** $\mathcal{O}(1)$
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```
if |A| = |B| = 1 then                         O(1)
    return a_0 · b_0                          O(1)
split A into A_0 and A_1                      O(n)
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return $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	$\mathcal{O}(n)$

We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) \ .$$

Example: Multiplying Two Integers

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

- Case 1: $f(n) = \mathcal{O}(n^{\log_b a-\epsilon})$ $\quad T(n) = \Theta(n^{\log_b a})$
- Case 2: $f(n) = \Theta(n^{\log_b a}\log^k n)$ $\quad T(n) = \Theta(n^{\log_b a}\log^{k+1} n)$
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In our case $a = 4$, $b = 2$, and $f(n) = \Theta(n)$. Hence, we are in Case 1, since $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$.

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We get a running time of $\mathcal{O}(n^2)$ for our algorithm.

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$\Rightarrow$ Not better then the “school method”.

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We can use the following identity to compute Z_1:

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Z_1 &= A_1B_0 + A_0B_1 \\
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1: **if** $|A| = |B| = 1$ **then**
2: **return** $a_0 \cdot b_0$
3: split A into A_0 and A_1
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5: $Z_2 \leftarrow \mathrm{mult}(A_1, B_1)$
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Hence,

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if $\lvert A\rvert = \lvert B\rvert = 1$ **then**	$\mathcal{O}(1)$
return $a_0 \cdot b_0$	$\mathcal{O}(1)$
split A into A_0 and A_1	$\mathcal{O}(n)$
split B into B_0 and B_1	$\mathcal{O}(n)$
$Z_2 \leftarrow \mathrm{mult}(A_1, B_1)$	$T(\frac{n}{2})$
$Z_0 \leftarrow \mathrm{mult}(A_0, B_0)$	$T(\frac{n}{2})$
$Z_1 \leftarrow \mathrm{mult}(A_0 + A_1, B_0 + B_1) - Z_2 - Z_0$	
return $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	

Example: Multiplying Two Integers

We can use the following identity to compute Z_1:

$$\begin{aligned} Z_1 &= A_1B_0 + A_0B_1 \\ &= (A_0 + A_1)\cdot(B_0 + B_1) - \overbrace{A_1B_1}^{=Z_2} - \overbrace{A_0B_0}^{=Z_0} \end{aligned}$$

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7: $Z_1 \leftarrow \mathrm{mult}(A_0 + A_1, B_0 + B_1) - Z_2 - Z_0$	$T(\frac{n}{2}) + \mathcal{O}(n)$
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Example: Multiplying Two Integers

We get the following recurrence:

$$T(n) = 3T\left(\frac{n}{2}\right) + \mathcal{O}(n) \ .$$

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

- Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $\quad T(n) = \Theta(n^{\log_b a})$
- Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$ $\quad T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
- Case 3: $f(n) = \Omega(n^{\log_b a + \epsilon})$ $\quad T(n) = \Theta(f(n))$

Again we are in Case 1. We get a running time of $\Theta(n^{\log_2 3}) \approx \Theta(n^{1.59})$.

A huge improvement over the "school method".

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6.3 The Characteristic Polynomial

Consider the recurrence relation:

$$c_0 T(n) + c_1 T(n-1) + c_2 T(n-2) + \cdots + c_k T(n-k) = f(n)$$

This is the general form of a linear recurrence relation of order k with constant coefficients ($c_0, c_k \neq 0$).

- $T(n)$ only depends on the k preceding values. This means the recurrence relation is of order k.
- The recurrence is linear as there are no products of $T[n]$'s.
- If $f(n) = 0$ then the recurrence relation becomes a linear, [illegible] recurrence relation of order k.

Note that we ignore boundary conditions for the moment.

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Observations:

- The solution $T[1], T[2], T[3], \ldots$ is completely determined by a set of boundary conditions that specify values for $T[1], \ldots, T[k]$.
- In fact, any k consecutive values completely determine the solution.
- k non-concecutive values might not be an appropriate set of boundary conditions (depends on the problem).

Approach:

- First determine all solutions that satisfy recurrence relation.
- Then pick the right one by analyzing boundary conditions.
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The Homogenous Case

The solution space

$$S = \Big\{ \mathcal{T} = T[1], T[2], T[3], \ldots \;\Big|\; \mathcal{T} \text{ fulfills recurrence relation} \Big\}$$

is a vector space. This means that if $\mathcal{T}_1, \mathcal{T}_2 \in S$, then also $\alpha\mathcal{T}_1 + \beta\mathcal{T}_2 \in S$, for arbitrary constants α, β.

How do we find a non-trivial solution?

We guess that the solution is of the form λ^n, $\lambda \neq 0$, and see what happens. In order for this guess to fulfill the recurrence we need

$$c_0\lambda^n + c_1\lambda^{n-1} + c_2 \cdot \lambda^{n-2} + \cdots + c_k \cdot \lambda^{n-k} = 0$$

for all $n \geq k$.

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Dividing by λ^{n-k} gives that all these constraints are identical to

$$c_0\lambda^k + c_1\lambda^{k-1} + c_2 \cdot \lambda^{k-2} + \cdots + c_k = 0$$

This means that if λ_i is a root (Nullstelle) of $P[\lambda]$ then $T[n] = \lambda_i^n$ is a solution to the recurrence relation.

Let $\lambda_1, \ldots, \lambda_k$ be the k (complex) roots of $P[\lambda]$. Then, because of the vector space property

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The Homogenous Case

Lemma 5

Assume that the characteristic polynomial has k distinct roots $\lambda_1, \dots, \lambda_k$. Then all solutions to the recurrence relation are of the form

$$\alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n + \cdots + \alpha_k \lambda_k^n \ .$$

Proof.

There is one solution for every possible choice of boundary conditions for $T[1], \dots, T[k]$.

We show that the above set of solutions contains one solution for every choice of boundary conditions.

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Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha_i's$ such that these conditions are met:

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We show that the column vectors are linearly independent. Then the above equation has a solution.

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} =$$

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} = \prod_{i=1}^{k} \lambda_i \cdot \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_{k-1}^{k-1} & \lambda_k^{k-1} \end{vmatrix}$$

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$$= \prod_{i=1}^{k} \lambda_i \cdot \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{k-2} & \lambda_1^{k-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{k-2} & \lambda_2^{k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-2} & \lambda_k^{k-1} \end{vmatrix}$$

Computing the Determinant

$$
\begin{vmatrix}
1 & \lambda_1 & \cdots & \lambda_1^{k-2} & \lambda_1^{k-1} \\
1 & \lambda_2 & \cdots & \lambda_2^{k-2} & \lambda_2^{k-1} \\
\vdots & \vdots & & \vdots & \vdots \\
1 & \lambda_k & \cdots & \lambda_k^{k-2} & \lambda_k^{k-1}
\end{vmatrix} =
$$

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$$\begin{vmatrix} 1 & \lambda_1 - \lambda_1 \cdot 1 & \cdots & \lambda_1^{k-2} - \lambda_1 \cdot \lambda_1^{k-3} & \lambda_1^{k-1} - \lambda_1 \cdot \lambda_1^{k-2} \\ 1 & \lambda_2 - \lambda_1 \cdot 1 & \cdots & \lambda_2^{k-2} - \lambda_1 \cdot \lambda_2^{k-3} & \lambda_2^{k-1} - \lambda_1 \cdot \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k - \lambda_1 \cdot 1 & \cdots & \lambda_k^{k-2} - \lambda_1 \cdot \lambda_k^{k-3} & \lambda_k^{k-1} - \lambda_1 \cdot \lambda_k^{k-2} \end{vmatrix}$$

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$$\begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & (\lambda_2 - \lambda_1) \cdot 1 & \cdots & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-3} & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & (\lambda_k - \lambda_1) \cdot 1 & \cdots & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-3} & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-2} \end{vmatrix}$$

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$$\prod_{i=2}^{k}(\lambda_i-\lambda_1)\cdot\begin{vmatrix} 1 & \lambda_2 & \cdots & \lambda_2^{k-3} & \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-3} & \lambda_k^{k-2} \end{vmatrix}$$

Computing the Determinant

Repeating the above steps gives:

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} = \prod_{i=1}^{k} \lambda_i \cdot \prod_{i>\ell} (\lambda_i - \lambda_\ell)$$

Hence, if all λ_i's are different, then the determinant is non-zero.

The Homogeneous Case

What happens if the roots are not all distinct?

Suppose we have a root λ_i with multiplicity (Vielfachheit) at least 2. Then not only is λ_i^n a solution to the recurrence but also $n\lambda_i^n$.

To see this consider the polynomial

$$P[\lambda] \cdot \lambda^{n-k} = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_k\lambda^{n-k}$$

Since λ_i is a root we can write this as $Q[\lambda] \cdot (\lambda - \lambda_i)^2$. Calculating the derivative gives a polynomial that still has root λ_i.

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This means

$$c_0 n\lambda_i^{n-1} + c_1(n-1)\lambda_i^{n-2} + \cdots + c_k(n-k)\lambda_i^{n-k-1} = 0$$

Hence,

$$c_0 \underbrace{n\lambda_i^{n}}_{T[n]} + c_1 \underbrace{(n-1)\lambda_i^{n-1}}_{T[n-1]} + \cdots + c_k \underbrace{(n-k)\lambda_i^{n-k}}_{T[n-k]} = 0$$

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$$c_0 n\lambda_i^n + c_1(n-1)\lambda_i^{n-1} + \cdots + c_k(n-k)\lambda_i^{n-k} = 0$$

(after taking the derivative; multiplying with λ; plugging in λ_i)

Doing this again gives

$$c_0 n^2\lambda_i^n + c_1(n-1)^2\lambda_i^{n-1} + \cdots + c_k(n-k)^2\lambda_i^{n-k} = 0$$

We can continue $j-1$ times.

Hence, $n^\ell\lambda_i^n$ is a solution for $\ell \in 0,\ldots,j-1$.

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Hence, $n^\ell\lambda_i^n$ is a solution for $\ell \in 0, \ldots, j-1$.

The Homogeneous Case

Lemma 6

Let $P[\lambda]$ denote the characteristic polynomial to the recurrence

$$c_0T[n] + c_1T[n-1] + \cdots + c_kT[n-k] = 0$$

Let λ_i, $i = 1,\ldots,m$ be the (complex) roots of $P[\lambda]$ with multiplicities ℓ_i. Then the general solution to the recurrence is given by

$$T[n] = \sum_{i=1}^{m}\sum_{j=0}^{\ell_i-1} \alpha_{ij}\cdot(n^j\lambda_i^n) \ .$$

The full proof is omitted. We have only shown that any choice of α_{ij}'s is a solution to the recurrence.

Example: Fibonacci Sequence

$$T[0] = 0$$
$$T[1] = 1$$
$$T[n] = T[n-1] + T[n-2] \text{ for } n \geq 2$$

The characteristic polynomial is

$$\lambda^2 - \lambda - 1$$

Finding the roots, gives

$$\lambda_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1}{2}\left(1 \pm \sqrt{5}\right)$$

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$$\alpha\left(\frac{1+\sqrt{5}}{2}\right)+\beta\left(\frac{1-\sqrt{5}}{2}\right)=1\implies\alpha-\beta=\frac{2}{\sqrt{5}}$$

Example: Fibonacci Sequence

Hence, the solution is

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

The Inhomogeneous Case

Consider the recurrence relation:

$$c_0T(n) + c_1T(n-1) + c_2T(n-2) + \cdots + c_kT(n-k) = f(n)$$

with $f(n) \neq 0$.

While we have a fairly general technique for solving homogeneous, linear recurrence relations the inhomogeneous case is different.

The Inhomogeneous Case

The general solution of the recurrence relation is

$$T(n) = T_h(n) + T_p(n) \ ,$$

where T_h is any solution to the homogeneous equation, and T_p is one particular solution to the inhomogeneous equation.

There is no general method to find a particular solution.

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The Inhomogeneous Case

Example:

$$T[n] = T[n-1] + 1 \qquad T[0] = 1$$

Then,

$$T[n-1] = T[n-2] + 1 \qquad (n \geq 2)$$

Subtracting the first from the second equation gives,

$$T[n] - T[n-1] = T[n-1] - T[n-2] \qquad (n \geq 2)$$

or

$$T[n] = 2T[n-1] - T[n-2] \qquad (n \geq 2)$$

I get a completely determined recurrence if I add $T[0] = 1$ and $T[1] = 2$.

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$T[1] = 2$ gives $1 + \beta = 2 \Longrightarrow \beta = 1$.

The Inhomogeneous Case

If $f(n)$ is a polynomial of degree r this method can be applied $r+1$ times to obtain a homogeneous equation:

$$T[n] = T[n-1] + n^2$$

Shift:

$$T[n-1] = T[n-2] + (n-1)^2 = T[n-2] + n^2 - 2n + 1$$

Difference:

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$$T[n] - T[n-1] = T[n-1] - T[n-2] + 2n - 1$$

$$T[n] = 2T[n-1] - T[n-2] + 2n - 1$$

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Shift:

$$T[n-1] = 2T[n-2] - T[n-3] + 2(n-1) - 1$$

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Shift:

$$\begin{aligned} T[n-1] &= 2T[n-2] - T[n-3] + 2(n-1) - 1 \\ &= 2T[n-2] - T[n-3] + 2n - 3 \end{aligned}$$

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and so on...

6.4 Generating Functions

Definition 7 (Generating Function)

Let $(a_n)_{n\geq 0}$ be a sequence. The corresponding

- generating function (Erzeugendenfunktion) is

$$F(z) := \sum_{n\geq 0} a_n z^n;$$

- exponential generating function (exponentielle Erzeugendenfunktion) is

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1. The generating function of the sequence $(1, 0, 0, \ldots)$ is

$$F(z) = 1\,.$$

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6.4 Generating Functions

There are two different views:

A generating function is a formal power series (formale Potenzreihe).

Then the generating function is an algebraic object.

Let $f = \sum_{n\geq 0} a_n z^n$ and $g = \sum_{n\geq 0} b_n z^n$.

- **Equality:** f and g are equal if $a_n = b_n$ for all n.
- **Addition:** $f + g := \sum_{n\geq 0}(a_n + b_n)z^n$.
- **Multiplication:** $f \cdot g := \sum_{n\geq 0} c_n z^n$ with $c = \sum_{p=0}^{n} a_p b_{n-p}$.

There are no convergence issues here.

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The arithmetic view:

We view a power series as a function $f : \mathbb{C} \to \mathbb{C}$.

Then, it is important to think about convergence/convergence radius etc.

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What does $\sum_{n\ge 0} z^n = \frac{1}{1-z}$ mean in the algebraic view?

It means that the power series $1 - z$ and the power series $\sum_{n\ge 0} z^n$ are invers, i.e.,

$$\left(1 - z\right) \cdot \left(\sum_{n\ge 0}^{\infty} z^n\right) = 1 \ .$$

This is well-defined.

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Hence, the generating function of the sequence $a_n = n + 1$ is $1/(1-z)^2$.

6.4 Generating Functions

We can repeat this

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Hence, the generating function of the sequence $a_n = (n+1)(n+2)$ is $\frac{2}{(1-z)^3}$.

6.4 Generating Functions

Computing the k-th derivative of $\sum z^n$.

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Hence:

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The generating function of the sequence $a_n = \binom{n+k}{k}$ is $\frac{1}{(1-z)^{k+1}}$.

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$$\sum_{n\geq 0} nz^n = \sum_{n\geq 0}(n+1)z^n - \sum_{n\geq 0} z^n$$

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The generating function of the sequence $a_n = n$ is $\frac{z}{(1-z)^2}$.

6.4 Generating Functions

We know

$$\sum_{n\geq 0} y^n = \frac{1}{1-y}$$

Hence,

$$\sum_{n\geq 0} a^n z^n = \frac{1}{1-az}$$

The generating function of the sequence $f_n = a^n$ is $\frac{1}{1-az}$.

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Example: $a_n = a_{n-1} + 1$, $a_0 = 1$

Suppose we have the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

$$A(z)$$

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&= a_0 + \sum_{n\geq 1} (a_{n-1} + 1) z^n \\
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&= zA(z) + \sum_{n\geq 0} z^n \\
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Solving for $A(z)$ gives

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Hence, $a_n = n + 1$.

Some Generating Functions

n-th sequence element	*generating function*

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1	$\dfrac{1}{1-z}$

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$\binom{n+k}{k}$	$\frac{1}{(1-z)^{k+1}}$

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n	$\frac{z}{(1-z)^2}$

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a^n	$\frac{1}{1-az}$

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$\frac{1}{n!}$	e^z

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n-th sequence element	*generating function*

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n-th sequence element	generating function
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$\sum_{i=0}^{n} f_i g_{n-i}$	$F \cdot G$

Some Generating Functions

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$f_n + g_n$	$F + G$
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f_{n-k} $(n \geq k)$; 0 otw.	$z^k F$

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$f_n + g_n$	$F + G$
$\sum_{i=0}^{n} f_i g_{n-i}$	$F \cdot G$
$f_{n-k} \quad (n \ge k);\quad 0$ otw.	$z^k F$
$\sum_{i=0}^{n} f_i$	$\dfrac{F(z)}{1-z}$

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nf_n	$z\frac{\mathrm{d}F(z)}{\mathrm{d}z}$

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$\sum_{i=0}^{n} f_i$	$\dfrac{F(z)}{1-z}$
nf_n	$z\dfrac{\mathrm{d}F(z)}{\mathrm{d}z}$
$c^n f_n$	$F(cz)$

Solving Recursions with Generating Functions

1. Set $A(z) = \sum_{n\geq 0} a_n z^n$.

Solving Recursions with Generating Functions

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6. The coefficients of the resulting power series are the a_n.

Example: $a_n = 2a_{n-1}$, $a_0 = 1$

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&= 1 + 3z \sum_{n\ge 0} a_n z^n + \sum_{n\ge 0} n z^n \\
&= 1 + 3zA(z) + \frac{z}{(1-z)^2}
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$$A(z) = \frac{(1-z)^2 + z}{(1-3z)(1-z)^2} = \frac{z^2 - z + 1}{(1-3z)(1-z)^2}$$

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This gives

$$\begin{aligned} z^2 - z + 1 &= A(1-z)^2 + B(1-3z)(1-z) + C(1-3z) \\ &= A(1-2z+z^2) + B(1-4z+3z^2) + C(1-3z) \end{aligned}$$

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This leads to the following conditions:

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This leads to the following conditions:

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which gives

$$A = \frac{7}{4} \quad B = -\frac{1}{4} \quad C = -\frac{1}{2}$$

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&= \sum_{n \ge 0} \left(\frac{7}{4} \cdot 3^n - \frac{1}{4} - \frac{1}{2}(n+1) \right) z^n
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&= \sum_{n \ge 0} \left(\frac{7}{4} \cdot 3^n - \frac{1}{2} n - \frac{3}{4} \right) z^n
\end{aligned}
$$

6. This means $a_n = \frac{7}{4} 3^n - \frac{1}{2} n - \frac{3}{4}$.

6.5 Transformation of the Recurrence

Example 9

$$
\begin{aligned}
f_0 &= 1 \\
f_1 &= 2 \\
f_n &= f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2 \,.
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6.5 Transformation of the Recurrence

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$$f_n = 2^{F_n}$$

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&= 3^3 g_{k-3} + 3^2 2^{k-2} + 32^{k-1} + 2^k \\
&= 2^k \cdot \sum_{i=0}^{k} \left(\frac{3}{2}\right)^i
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&= 3\left[3g_{k-2} + 2^{k-1}\right] + 2^k \\
&= 3^2\left[g_{k-2}\right] + 32^{k-1} + 2^k \\
&= 3^2\left[3g_{k-3} + 2^{k-2}\right] + 32^{k-1} + 2^k \\
&= 3^3 g_{k-3} + 3^2 2^{k-2} + 32^{k-1} + 2^k \\
&= 2^k \cdot \sum_{i=0}^{k} \left(\frac{3}{2}\right)^i \\
&= 2^k \cdot \frac{(\frac{3}{2})^{k+1} - 1}{1/2}
\end{aligned}$$

6 Recurrences

We get

$$
\begin{aligned}
g_k &= 3\left[g_{k-1}\right] + 2^k \\
&= 3\left[3g_{k-2} + 2^{k-1}\right] + 2^k \\
&= 3^2\left[g_{k-2}\right] + 32^{k-1} + 2^k \\
&= 3^2\left[3g_{k-3} + 2^{k-2}\right] + 32^{k-1} + 2^k \\
&= 3^3 g_{k-3} + 3^2 2^{k-2} + 32^{k-1} + 2^k \\
&= 2^k \cdot \sum_{i=0}^{k} \left(\frac{3}{2}\right)^i \\
&= 2^k \cdot \frac{(\frac{3}{2})^{k+1} - 1}{1/2} = 3^{k+1} - 2^{k+1}
\end{aligned}
$$

6 Recurrences

Let $n = 2^k$:

$$g_k = 3^{k+1} - 2^{k+1}, \text{ hence}$$
$$f_n = 3 \cdot 3^k - 2 \cdot 2^k$$

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