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We want to solve the following linear program:

- $\min v = c^t x$ subject to $Ax = 0$ and $x \in \Delta$.
- Here $\Delta = \{x \in \mathbb{R}^n \mid e^t x = 1, x \geq 0\}$ with $e^t = (1, \ldots, 1)$ denotes the standard simplex in $\mathbb{R}^n$.

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1. A is an $m \times n$-matrix with rank m.
2. $Ae = 0$, i.e., the center of the simplex is feasible.
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The algorithm computes strictly feasible interior points $x^{(0)} = \frac{e}{n}, x^{(1)}, x^{(2)}, \dots$ with

$$c^t x^{(k)} \leq 2^{-\Theta(L)} c^t x^{(0)}$$

For $k = \Theta(L)$. A point x is strictly feasible if $x > 0$.

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Iteration:

1. Distort the problem by mapping the simplex onto itself so that the current point $\bar{x}$ moves to the center.
2. Project the optimization direction c onto the feasible region. Determine a distance to travel along this direction such that you do not leave the simplex (and you do not touch the border). $\hat{x}_{\text{new}}$ is the point you reached.
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The Transformation

Let $\bar{Y} = \mathrm{diag}(\bar{x})$ the diagonal matrix with entries $\bar{x}$ on the diagonal.

Define

$$F_{\bar{x}} : x \mapsto \frac{\bar{Y}^{-1}x}{e^t\bar{Y}^{-1}x} \enspace .$$

The inverse function is

$$F_{\bar{x}}^{-1} : \hat{x} \mapsto \frac{\bar{Y}\hat{x}}{e^t\bar{Y}\hat{x}} \enspace .$$

Note that $\bar{x} > 0$ in every coordinate. Therefore the above is well defined.

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Properties

$F_{\bar{x}}^{-1}$ really is the inverse of $F_{\bar{x}}$:

$$F_{\bar{x}}(F_{\bar{x}}^{-1}(\hat{x})) = \frac{\bar{Y}^{-1}\frac{\bar{Y}\hat{x}}{e^t\bar{Y}\hat{x}}}{e^t\bar{Y}^{-1}\frac{\bar{Y}\hat{x}}{e^t\bar{Y}\hat{x}}} = \frac{\hat{x}}{e^t\hat{x}} = \hat{x}$$

because $\hat{x} \in \Delta$.

Note that in particular every $\hat{x} \in \Delta$ has a preimage (Urbild) under $F_{\bar{x}}$.

Properties

$\bar{x}$ is mapped to e/n

$$F_{\bar{x}}(\bar{x}) = \frac{\bar{Y}^{-1}\bar{x}}{e^t\bar{Y}^{-1}\bar{x}} = \frac{e}{e^t e} = \frac{e}{n}$$

Properties

A unit vectors e_i is mapped to itself:

$$F_{\bar{x}}(e_i) = \frac{\bar{Y}^{-1}e_i}{e^t\bar{Y}^{-1}e_i} = \frac{(0,\dots,0,1/\bar{x}_i,0,\dots,0)^t}{e^t(0,\dots,0,1/\bar{x}_i,0,\dots,0)^t} = e_i$$

Properties

All nodes of the simplex are mapped to the simplex:

$$F_{\bar{x}}(x) = \frac{\bar{Y}^{-1}x}{e^t\bar{Y}^{-1}x} = \frac{\left(\frac{x_1}{\bar{x}_1}, \dots, \frac{x_n}{\bar{x}_n}\right)^t}{e^t\left(\frac{x_1}{\bar{x}_1}, \dots, \frac{x_n}{\bar{x}_n}\right)^t} = \frac{\left(\frac{x_1}{\bar{x}_1}, \dots, \frac{x_n}{\bar{x}_n}\right)^t}{\sum_i \frac{x_i}{\bar{x}_i}} \in \Delta$$

The Transformation

Easy to check:

- $F_{\bar{x}}^{-1}$ really is the inverse of $F_{\bar{x}}$.
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Since the optimum solution is 0 this problem is the same as

$$\min\{\hat{c}^t \hat{x} \mid \hat{A}\hat{x} = 0, \hat{x} \in \Delta\}$$

with $\hat{c} = \bar{Y}^t c = \bar{Y}c$ and $\hat{A} = A\bar{Y}$.

We still need to make e/n feasible.

- We know that our LP is feasible. Let $\bar{x}$ be a feasible point.
- Apply $F_{\bar{x}}$, and solve

$$\min\{\hat{c}^t x \mid \hat{A}x = 0; x \in \Delta\}$$

- The feasible point is moved to the center.

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When computing $\hat{x}_{\text{new}}$ we do not want to leave the simplex or touch its boundary (why?).

For this we compute the radius of a ball that completely lies in the simplex.

$$B\left(\frac{e}{n},\rho\right) = \left\{x \in \mathbb{R}^n \mid \left\|x - \frac{e}{n}\right\| \le \rho\right\} \ .$$

We are looking for the largest radius r such that

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This holds for $r = \|\frac{e}{n} - (e - e_1)\frac{1}{n-1}\|$. ($r$ is the distance between the center e/n and the center of the $(n-1)$-dimensional simplex obtained by intersecting a side ($x_i = 0$) of the unit cube with Δ.)

This gives $r = \frac{1}{\sqrt{n(n-1)}}$.

Now we consider the problem

$$\min\{\hat{c}^t x \mid \hat{A}x = 0, x \in B(e/n, r) \cap \Delta\}$$

This problem is easy to solve!!!

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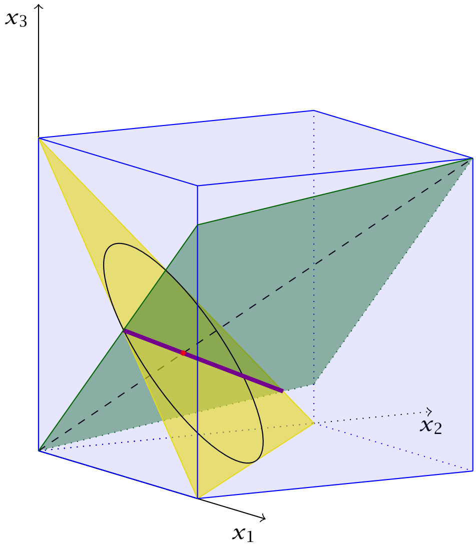
The Simplex



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Ideally we would like to go in direction of $-\hat{c}$ (starting from the center of the simplex).

However, doing this may violate constraints $\hat{A}\hat{x} = 0$ or the constraint $\hat{x} \in \Delta$.

Therefore we first project $\hat{c}$ on the nullspace of

$$B = \begin{pmatrix} \hat{A} \\ e^t \end{pmatrix}$$

We use

$$P = I - B^t(BB^t)^{-1}B$$

Then

$$\hat{d} = P\hat{c}$$

is the required projection.

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We get the new point

$$\hat{x}(\rho) = \frac{e}{n} - \rho \frac{\hat{d}}{\|\hat{d}\|}$$

for $\rho < r$.

Choose $\rho = \alpha r$ with $\alpha = 1/4$.

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Iteration of Karmarkars Algorithm

- Current solution $\bar{x}$. $\bar{Y} := \operatorname{diag}(\bar{x}_1, \ldots, \bar{x}_n)$.
- Transform problem via $F_{\bar{x}}(x) = \frac{\bar{Y}^{-1}x}{e^t \bar{Y}^{-1}x}$.
  Let $\hat{c} = \bar{Y}c$, and $\hat{A} = A\bar{Y}$.
- Compute
  $$\hat{d} = (I - B^t(BB^t)^{-1}B)\hat{c} \ ,$$
  where $B = \begin{pmatrix} \hat{A} \\ e^t \end{pmatrix}$.
- Set
  $$\hat{x}_{\text{new}} = \frac{e}{n} - \rho \frac{\hat{d}}{\|\hat{d}\|} \ ,$$
  with $\rho = \alpha r$ with $\alpha = 1/4$ and $r = 1/\sqrt{n(n-1)}$.
- Compute $\bar{x}_{\text{new}} = F_{\bar{x}}^{-1}(\hat{x}_{\text{new}})$.

Lemma 2

The new point $\hat{x}_{\text{new}}$ in the transformed space is the point that minimizes the cost $\hat{c}^t\hat{x}$ among all feasible points in $B(\frac{e}{n},\rho)$.

Proof: Let $\hat{z}$ be another feasible point in $B(\frac{e}{n}, \rho)$.

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which means that the cost-difference between $\hat{x}_{\text{new}}$ and $\hat{z}$ is the same measured w.r.t. the cost-vector $\hat{c}$ or the projected cost-vector $\hat{d}$.

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This gives $\hat{d}(\hat{x}_{\text{new}} - \hat{z}) \le 0$ and therefore $\hat{c}\hat{x}_{\text{new}} \le \hat{c}\hat{z}$.

In order to measure the progress of the algorithm we introduce a potential function f:

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- The function f is invariant to scaling (i.e., $f(kx) = f(x)$).
- The potential function essentially measures cost (note the term $n \ln(c^t x)$) but it penalizes us for choosing x_j values very small (by the term $-\sum_j \ln(x_j)$; note that $-\ln(x_j)$ is always positive).

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Observation:

This means the potential of a point in the transformed space is simply the potential of its pre-image under F.

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Observation:
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Note that if we are interested in potential-change we can ignore the additive term above. Then f and $\hat{f}$ have the same form; only c is replaced by $\hat{c}$.

The basic idea is to show that one iteration of Karmarkar results in a constant decrease of $\hat{f}$. This means

$$\hat{f}(\hat{x}_{\text{new}}) \le \hat{f}(\frac{e}{n}) - \delta \quad ,$$

where δ is a constant.

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Lemma 3

There is a feasible point z (i.e., $\hat{A}z = 0$) in $B(\frac{e}{n}, \rho) \cap \Delta$ that has

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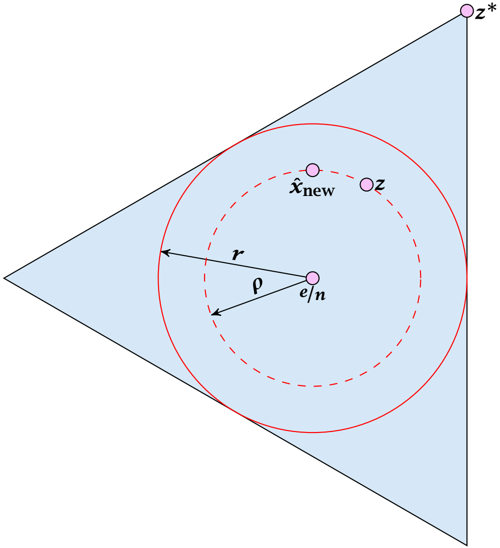
Note that this shows the existence of a good point within the ball. In general it will be difficult to find this point.


z^*
$\hat{x}_{\text{new}}$
z
r
ρ
e/n

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$$z = (1 - \lambda)\frac{e}{n} + \lambda z^*$$

for some positive $\lambda < 1$.

Hence,

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Therefore,

$$\frac{\hat{c}^t \frac{e}{n}}{\hat{c}^t z} = \frac{1}{1-\lambda}$$

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&= \sum_j \ln(\frac{n}{1-\lambda}((1-\lambda)\frac{1}{n} + \lambda z_j^*)) \\
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This gives the lemma.

Lemma 4

If we choose $\alpha = 1/4$ *and* $n \geq 4$ *in Karmarkars algorithm the point* $\hat{x}_{\text{new}}$ *satisfies*

$$\hat{f}(\hat{x}_{\text{new}}) \leq \hat{f}(\frac{e}{n}) - \delta$$

with $\delta = 1/10$*.*

Proof:

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\begin{aligned}
g(\hat{x}) &= n \ln \frac{\hat{c}^t \hat{x}}{\hat{c}^t \frac{e}{n}} \\
&= n(\ln \hat{c}^t \hat{x} - \ln \hat{c}^t \frac{e}{n}) \ .
\end{aligned}
$$

Proof:

Define

$$g(\hat{x}) = n \ln \frac{\hat{c}^t \hat{x}}{\hat{c}^t \frac{e}{n}} = n(\ln \hat{c}^t \hat{x} - \ln \hat{c}^t \frac{e}{n}) \ .$$

This is the change in the cost part of the potential function when going from the center $\frac{e}{n}$ to the point $\hat{x}$ in the transformed space.

Similar, the penalty when going from $\frac{e}{n}$ to w increases by

$$h(\hat{x}) = \text{pen}(\hat{x}) - \text{pen}(\frac{e}{n}) = -\sum_j \ln \frac{\hat{x}_j}{\frac{1}{n}}$$

where $\text{pen}(v) = -\sum_j \ln(v_j)$.

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$$\begin{aligned}\hat{f}(\frac{e}{n}) - \hat{f}(\hat{x}_{\text{new}}) = {} & [\hat{f}(\frac{e}{n}) - \hat{f}(z)] \\ & + h(z) \\ & - h(\hat{x}_{\text{new}}) \\ & + [g(z) - g(\hat{x}_{\text{new}})]\end{aligned}$$

where z is the point in the ball where $\hat{f}$ achieves its minimum.

We want to derive a lower bound on

$$\hat{f}(\frac{e}{n}) - \hat{f}(\hat{x}_{\text{new}}) = [\hat{f}(\frac{e}{n}) - \hat{f}(z)] + h(z) - h(\hat{x}_{\text{new}}) + [g(z) - g(\hat{x}_{\text{new}})]$$

where z is the point in the ball where $\hat{f}$ achieves its minimum.

We have

$$[\hat{f}(\frac{e}{n}) - \hat{f}(z)] \geq \ln(1+\alpha)$$

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We have

$$[g(z) - g(\hat{x}_{\text{new}})] \geq 0$$

since $\hat{x}_{\text{new}}$ is the point with minimum cost in the ball, and g is monotonically increasing with cost.

We show that the change $h(w)$ in penalty when going from e/n to w fulfills

$$|h(w)| \le \frac{\beta^2}{2(1-\beta)}$$

where $\beta = n\alpha r$ and w is some point in the ball $B(\frac{e}{n}, \alpha r)$.

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Hence,

$$\hat{f}(\frac{e}{n}) - \hat{f}(\hat{x}_{\text{new}}) \geq \ln(1+\alpha) - \frac{\beta^2}{(1-\beta)} \ .$$

Lemma 5

For $|x| \leq \beta < 1$

$$|\ln(1+x) - x| \leq \frac{x^2}{2(1-\beta)} \enspace .$$

This gives for $w \in B(\frac{e}{n}, \rho)$

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\begin{aligned}
|h(w)| &= \left| \sum_j \ln \frac{w_j}{1/n} \right| \\
&= \left| \sum_j \ln \left(\frac{1/n + (w_j - 1/n)}{1/n} \right) - \sum_j n \left(w_j - \frac{1}{n} \right) \right|
\end{aligned}
$$

This gives for $w \in B(\frac{e}{n}, \rho)$

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&= \left| \sum_j \left[\ln\Big(1 + n(w_j - 1/n)\Big) - n(w_j - 1/n) \right] \right|
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&\le \frac{(\alpha n r)^2}{2(1 - \alpha n r)}
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$$

The decrease in potential is therefore at least

$$\ln(1+\alpha) - \frac{\beta^2}{1-\beta}$$

with $\beta = n\alpha r = \alpha\sqrt{\frac{n}{n-1}}$.

It can be shown that this is at least $\frac{1}{10}$ for $n \geq 4$ and $\alpha = 1/4$.

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Let $\bar{x}^{(k)}$ be the current point after the k-th iteration, and let $\bar{x}^{(0)} = \frac{e}{n}$.

Then $f(\bar{x}^{(k)}) \leq f(e/n) - k/10$.

This gives

$$n \ln \frac{c^t \bar{x}^{(k)}}{c^t \frac{e}{n}} \leq \sum_j \ln \bar{x}_j^{(k)} - \sum_j \ln \frac{1}{n} - k/10$$

$$\leq n \ln n - k/10$$

Choosing $k = 10n(\ell + \ln n)$ with $\ell = \Theta(L)$ we get

$$\frac{c^t \bar{x}^{(k)}}{c^t \frac{e}{n}} \leq e^{-\ell} \leq 2^{-\ell} .$$

Hence, $\Theta(nL)$ iterations are sufficient. One iteration can be performed in time $\mathcal{O}(n^3)$.

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