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Randomized Algorithms

Exercise Sheet 1

Due: October 20, 2014

Exercise 1 (5 points)
Consider the set of integers $A_n = \{1, 2, \ldots, n\}$. A permutation π of A_n can be represented as a one-to-one function $f_\pi : A_n \to A_n$, where $f_\pi(i)$ is the position of the integer i in the permutation π. For a given permutation π of A_n, we say that i is a *fixed point* of π if $f_\pi(i) = i$.

- What is the expected number of fixed points in a permutation chosen uniformly at random among all possible $n!$ permutations?

Exercise 2 (5 points)
One way to pick a permutation π of the set of integers $A_n = \{1, 2, \ldots, n\}$ is the following:

- Run Randomized Quicksort.
- Let T be the tree corresponding to the execution of Randomized Quicksort.
- Perform a level-order traversal of T by visiting its nodes in increasing order of level numbers and in a left-to-right order within each level.

Find an example which shows that π is not uniformly distributed over the space of all permutations of the elements in A_n.

Exercise 3 (10 points)
Let $a_1, a_2, \ldots, a_n$ be a list of distinct numbers. We say that a_i and a_j are *inverted* if $i < j$ and $a_i > a_j$. *Bubblesort* algorithm swaps pairwise adjacent numbers a_i, a_{i+1} until there are no more inversions. Suppose that the input to Bubblesort is a permutation chosen uniformly at random among all possible $n!$ permutations. What is the expected number of inversions needed to be corrected by Bubblesort?

Exercise 4 (10 points)
In the min (s,t)-cut problem, we are given an undirected connected multigraph $G = (V, E)$ with two distinguished vertices $s, t \in V$. An (s,t)-cut is a subset of edges $C \subseteq E$ whose removal from G disconnects s from t. The goal is to find an (s,t)-cut of minimum size. Consider the following adaptation of the min cut algorithm presented in class for the min (s,t)-cut problem. The algorithm contracts edges step by step. As the algorithm proceeds, the vertices s and t may be replaced by new vertices as a result edge contractions. However, we ensure that s and t are not in the same vertex at any step of the algorithm.

- Find an example of a graph G in which the probability that the algorithm finds a min (s,t)-cut is at most $(\frac{2}{3})^n$, where n in the number of vertices in G.

Exercise 5 (10 points)
We are given a biased coin which lands heads with probability p, $0 < p < \frac{1}{2}$. How can we use this coin in order to generate an unbiased coin-flip, i.e. such that $Pr(heads) = Pr(tails) = \frac{1}{2}$?

Hint: Consider two consecutive flips of the biased coin.