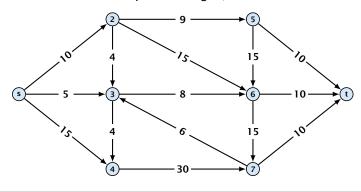
10 Introduction

Flow Network

- directed graph G = (V, E); edge capacities c(e)
- two special nodes: source s; target t;
- ▶ no edges entering *s* or leaving *t*;
- at least for now: no parallel edges;



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Cuts

Definition 1

An (s, t)-cut in the graph G is given by a set $A \subset V$ with $s \in A$ and $t \in V \setminus A$.

Definition 2

The capacity of a cut A is defined as

$$\operatorname{cap}(A, V \setminus A) := \sum_{e \in \operatorname{out}(A)} c(e) ,$$

where $\operatorname{out}(A)$ denotes the set of edges of the form $A \times V \setminus A$ (i.e. edges leaving A).

Minimum Cut Problem: Find an (s, t)-cut with minimum capacity.

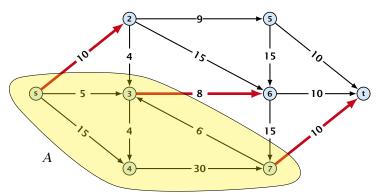
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Cuts

Example 3



The capacity of the cut is $cap(A, V \setminus A) = 28$.

Flows

Definition 4

An (s,t)-flow is a function $f: E \mapsto \mathbb{R}^+$ that satisfies

1. For each edge *e*

$$0 \le f(e) \le c(e)$$
.

(capacity constraints)

2. For each $v \in V \setminus \{s, t\}$

$$\sum_{e \in \text{out}(v)} f(e) = \sum_{e \in \text{into}(v)} f(e) .$$

(flow conservation constraints)

Flows

Definition 5

The value of an (s, t)-flow f is defined as

$$val(f) = \sum_{e \in out(s)} f(e)$$
.

Maximum Flow Problem: Find an (s, t)-flow with maximum value.

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Flows

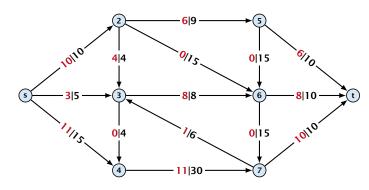
Lemma 7 (Flow value lemma)

Let f be a flow, and let $A \subseteq V$ be an (s,t)-cut. Then the net-flow across the cut is equal to the amount of flow leaving s, i.e.,

$$\operatorname{val}(f) = \sum_{e \in \operatorname{out}(A)} f(e) - \sum_{e \in \operatorname{into}(A)} f(e)$$
.

Flows

Example 6



The value of the flow is val(f) = 24.

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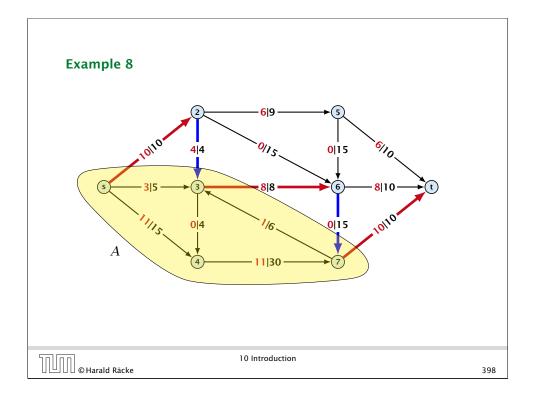
Proof.

$$val(f) = \sum_{e \in out(s)} f(e)$$

$$= \sum_{e \in out(s)} f(e) + \sum_{v \in A \setminus \{s\}} \left(\sum_{e \in out(v)} f(e) - \sum_{e \in in(v)} f(e) \right)$$

$$= \sum_{e \in out(A)} f(e) - \sum_{e \in into(A)} f(e)$$

The last equality holds since every edge with both end-points in A contributes negatively as well as positively to the sum in Line 2. The only edges whose contribution doesn't cancel out are edges leaving or entering A.



Corollary 9

Let f be an (s,t)-flow and let A be an (s,t)-cut, such that

$$val(f) = cap(A, V \setminus A).$$

Then f is a maximum flow.

Proof.

Suppose that there is a flow f^{\prime} with larger value. Then

$$cap(A, V \setminus A) < val(f')$$

$$= \sum_{e \in out(A)} f'(e) - \sum_{e \in into(A)} f'(e)$$

$$\leq \sum_{e \in out(A)} f'(e)$$

$$\leq cap(A, V \setminus A)$$

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