

Example: $a_n = 3a_{n-1} + n, a_0 = 1$

5. Write $f(z)$ as a formal power series:

This leads to the following conditions:

$$\begin{aligned}A + B + C &= 1 \\2A + 4B + 3C &= 1 \\A + 3B &= 1\end{aligned}$$

which gives

$$A = \frac{7}{4} \quad B = -\frac{1}{4} \quad C = -\frac{1}{2}$$

Example: $a_n = 3a_{n-1} + n, a_0 = 1$

5. Write $f(z)$ as a formal power series:

$$\begin{aligned}A(z) &= \frac{7}{4} \cdot \frac{1}{1-3z} - \frac{1}{4} \cdot \frac{1}{1-z} - \frac{1}{2} \cdot \frac{1}{(1-z)^2} \\&= \frac{7}{4} \cdot \sum_{n \geq 0} 3^n z^n - \frac{1}{4} \cdot \sum_{n \geq 0} z^n - \frac{1}{2} \cdot \sum_{n \geq 0} (n+1)z^n \\&= \sum_{n \geq 0} \left(\frac{7}{4} \cdot 3^n - \frac{1}{4} - \frac{1}{2}(n+1) \right) z^n \\&= \sum_{n \geq 0} \left(\frac{7}{4} \cdot 3^n - \frac{1}{2}n - \frac{3}{4} \right) z^n\end{aligned}$$

6. This means $a_n = \frac{7}{4}3^n - \frac{1}{2}n - \frac{3}{4}$.

6.5 Transformation of the Recurrence

Example 6

$$\begin{aligned}f_0 &= 1 \\f_1 &= 2 \\f_n &= f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2.\end{aligned}$$

Define

$$g_n := \log f_n.$$

Then

$$\begin{aligned}g_n &= g_{n-1} + g_{n-2} \text{ for } n \geq 2 \\g_1 &= \log 2 = 1 (\text{for } \log = \log_2), \quad g_0 = 0 \\g_n &= F_n \text{ (} n\text{-th Fibonacci number)} \\f_n &= 2^{F_n}\end{aligned}$$

6.5 Transformation of the Recurrence

Example 7

$$\begin{aligned}f_1 &= 1 \\f_n &= 3f_{\frac{n}{2}} + n; \text{ for } n = 2^k, k \geq 1;\end{aligned}$$

Define

$$g_k := f_{2^k}.$$

Then:

$$\begin{aligned}g_0 &= 1 \\g_k &= 3g_{k-1} + 2^k, \quad k \geq 1\end{aligned}$$

6 Recurrences

We get

$$\begin{aligned}g_k &= 3[g_{k-1}] + 2^k \\&= 3[3g_{k-2} + 2^{k-1}] + 2^k \\&= 3^2[g_{k-2}] + 3 \cdot 2^{k-1} + 2^k \\&= 3^2[3g_{k-3} + 2^{k-2}] + 3 \cdot 2^{k-1} + 2^k \\&= 3^3g_{k-3} + 3^2 \cdot 2^{k-2} + 3 \cdot 2^{k-1} + 2^k \\&= 2^k \cdot \sum_{i=0}^k \left(\frac{3}{2}\right)^i \\&= 2^k \cdot \frac{\left(\frac{3}{2}\right)^{k+1} - 1}{1/2} = 3^{k+1} - 2^{k+1}\end{aligned}$$

6 Recurrences

Let $n = 2^k$:

$$\begin{aligned}g_k &= 3^{k+1} - 2^{k+1}, \text{ hence} \\f_n &= 3 \cdot 3^k - 2 \cdot 2^k \\&= 3(2^{\log 3})^k - 2 \cdot 2^k \\&= 3(2^k)^{\log 3} - 2 \cdot 2^k \\&= 3n^{\log 3} - 2n.\end{aligned}$$

6 Recurrences

Bibliography

- [MS08] Kurt Mehlhorn, Peter Sanders:
Algorithms and Data Structures — The Basic Toolbox,
Springer, 2008
- [CLRS90] Thomas H. Cormen, Charles E. Leiserson, Ron L. Rivest, Clifford Stein:
Introduction to algorithms (3rd ed.),
MIT Press and McGraw-Hill, 2009
- [Liu85] Chung Laung Liu:
Elements of Discrete Mathematics
McGraw-Hill, 1985

The Karatsuba method can be found in [MS08] Chapter 1. Chapter 4.3 of [CLRS90] covers the "Substitution method" which roughly corresponds to "Guessing+induction". Chapters 4.4, 4.5, 4.6 of this book cover the master theorem. Methods using the characteristic polynomial and generating functions can be found in [Liu85] Chapter 10.