Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

5. Write f(z) as a formal power series:

This leads to the following conditions:

$$A + B + C = 1$$
$$2A + 4B + 3C = 1$$
$$A + 3B = 1$$

which gives

$$A = \frac{7}{4}$$
 $B = -\frac{1}{4}$ $C = -\frac{1}{2}$

113

115

6.4 Generating Functions

6.5 Transformation of the Recurrence Example 6 $f_0 = 1$ $f_1 = 2$ $f_n = f_{n-1} \cdot f_{n-2}$ for $n \ge 2$. Define $g_n := \log f_n$.

Then

$$g_n = g_{n-1} + g_{n-2}$$
 for $n \ge 2$
 $g_1 = \log 2 = 1$ (for $\log = \log_2$), $g_0 = 0$
 $g_n = F_n$ (*n*-th Fibonacci number)
 $f_n = 2^{F_n}$

© Harald Räcke

6.5 Transformation of the Recurrence

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

5. Write f(z) as a formal power series:

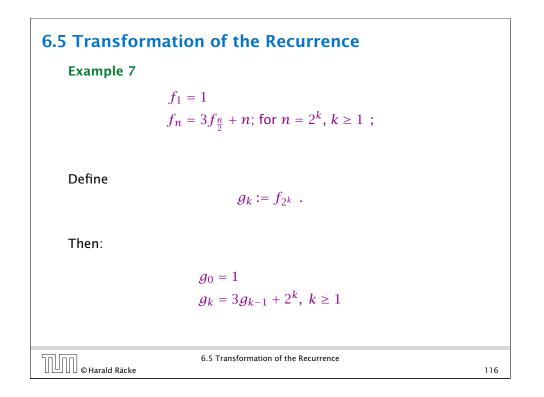
$$A(z) = \frac{7}{4} \cdot \frac{1}{1-3z} - \frac{1}{4} \cdot \frac{1}{1-z} - \frac{1}{2} \cdot \frac{1}{(1-z)^2}$$

$$= \frac{7}{4} \cdot \sum_{n \ge 0} 3^n z^n - \frac{1}{4} \cdot \sum_{n \ge 0} z^n - \frac{1}{2} \cdot \sum_{n \ge 0} (n+1) z^n$$

$$= \sum_{n \ge 0} \left(\frac{7}{4} \cdot 3^n - \frac{1}{4} - \frac{1}{2}(n+1)\right) z^n$$

$$= \sum_{n \ge 0} \left(\frac{7}{4} \cdot 3^n - \frac{1}{2}n - \frac{3}{4}\right) z^n$$

6. This means $a_n = \frac{7}{4}3^n - \frac{1}{2}n - \frac{3}{4}$.



6 Recurrences

We get

$$g_{k} = 3 [g_{k-1}] + 2^{k}$$

$$= 3 [3g_{k-2} + 2^{k-1}] + 2^{k}$$

$$= 3^{2} [g_{k-2}] + 32^{k-1} + 2^{k}$$

$$= 3^{2} [3g_{k-3} + 2^{k-2}] + 32^{k-1} + 2^{k}$$

$$= 3^{3}g_{k-3} + 3^{2}2^{k-2} + 32^{k-1} + 2^{k}$$

$$= 2^{k} \cdot \sum_{i=0}^{k} \left(\frac{3}{2}\right)^{i}$$

$$= 2^{k} \cdot \frac{\left(\frac{3}{2}\right)^{k+1} - 1}{1/2} = 3^{k+1} - 2^{k+1}$$
6.5 Transformation of the Recurrence

6 Recurrences

Let $n = 2^k$:

$$g_k = 3^{k+1} - 2^{k+1}, \text{ hence}$$

$$f_n = 3 \cdot 3^k - 2 \cdot 2^k$$

$$= 3(2^{\log 3})^k - 2 \cdot 2^k$$

$$= 3(2^k)^{\log 3} - 2 \cdot 2^k$$

$$= 3n^{\log 3} - 2n .$$

G Harald Räcke

6.5 Transformation of the Recurrence

118

